Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 32, 2018

THE PROCESS OF CUTTING GML_m^n BODIES WITH d_m-KNIVES *

Ilia Tavkhelidze Johan Gielis

Abstract. In this article, we will represent Generalized Möbius Listing's bodies $GML_m^n\{\nu,\Theta\}$ as elements of a permutation group. Also, we will give a definition of d_m -knife it analytical representation. Finally, we show that a full cutting of this $GML_m^n\{\nu,\Theta\}$ body can be considered as an action of a "d_m -knife".

Keywords and phrases: Analytic representation, toroidal lines, Generalized Möbius-Listing's bodies.

AMS subject classification (2010): 51N20, 51N25.

In earlier works, a definition and an analytical representation of GML bodies were given (see f.e. [1-3]). Therefore, we just recall:

Definition 1. A Generalized Möbius Listing's body - $GML_m^n\{\nu,\Theta\}$ - is obtained by identifying the opposite ends of the prism $PR_m \equiv A_0A_1\cdots A_{m-1}A'_0A'_1\cdots A'_{m-1}$ with the central axis OO' in such a way that:

A) For any integer $n \in Z$ and $i = 0,1,\ldots,m-1$, each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge A_iA_{i+1} (each $i+1$ is number of edge of initial prism) coincides with the edge

$$A'_{i+n}A'_{i+n+1} \equiv A'_{mod_m(i+n)}A'_{mod_m(i+n+1)};$$

B) The integer $n \in Z$ denotes the number of rotations of the end of the prism with respect to the axis OO' before the identification, with $n \equiv \omega m + \kappa$; $\omega \in Z$ and $\kappa = 0,1,2,\ldots,m-1$. If $n > 0$, the rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise;

C) The original axis OO' of the prism is transformed in a closed space (or plane) curve Θ, which is called the basic line with characteristic ν. It is defined by the function $R(\theta)$ in (1). One form of analytical representation of GML bodies is formula (1.2.1*) in [2].

$$\begin{aligned} X(\tau,\psi,\theta) &= [R(\theta) + p(x,z)\cos(\mu g(\theta))]\cos(\theta),\\ Y(\tau,z,\theta) &= [R(\theta) + p(x,z)\cos(\mu g(\theta))]\sin(\theta), \qquad (1)\\ Z(x,z,\theta) &= p(x,z)\sin(\mu g(\theta)). \end{aligned}$$

*The authors are deeply grateful to Prof. Paolo Emilio Ricci for discussion. The project partially has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR/358/5-109/14).

Remark 1. The cross section of the prism or the GML can be a regular polygon with m vertices or a circle with m equally spaced points. Using Gielis transformations, one can be transformed into the other.

Remark 2. Each Generalized Möbius Listing's body $GML_m^n\{\nu,\Theta\}$ can be considered as a geometric representation of the element of the permutation group, in particular

$$\begin{pmatrix} A_0A_1 & A_1A_2 & \cdots & A_{i-1}A_i & \cdots & A_{m-2}A_{m-1} & A_{m-1}A_0 \\ A'_nA'_{n+1} & A'_{n+1}A'_{n+2} & \cdots & A'_{n+i-1}A'_{n+i} & \cdots & A'_{n+m-2}A'_{n+m-1} & A'_{n+m-1}A'_n \end{pmatrix} \equiv$$
$$\begin{pmatrix} 1 & 2 & \cdots & i & \cdots & m-1 & m \\ k & k+1 & \cdots & \mathrm{mod}_m(k+i) & \cdots & \mathrm{mod}_m(k+m-1) & \mathrm{mod}_m(k+m) \end{pmatrix}. \tag{2}$$

This follows from the facts that 1. the sub-indices change cyclically from 1 to $m-1$, and 2. the sides are identified by Definition 1. The first row of (2) gives the numbering in the original prism and the second row gives the corresponding numbering after rotation and identification according to definition of GML.

Remark 3. These permutations form a subgroup of the group of permutations, in which:

• The number of elements of this subgroup equals $\boldsymbol{m}!$;

• The numbering in both rows stricly follows the usual sequence of numbers (in row 1 from 1 to m;

• Each element, except the neutral element, has a certain number of cycles and geometrically, this indicates in - how many different colors the surface of a given body can be painted, without lifting the brush and not passing the ribs. Some examples are given in Table 1.

Permutation	Cycles or color	Geometric meaning
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	four	$GML_4^{\omega 4}\{\nu,\Theta\}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	one	$GML_4^{\omega 4+1}\{\nu,\Theta\}$ or $GML_4^{\omega 4+3}\{\nu,\Theta\}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	two	$GML_4^{\omega 4+2}\{\nu,\Theta\}$

Table 1. Permutations and GML bodies

Corollary 1. *Each $GML_m^n\{\nu,\Theta\}$ body has a j-colored surface where $j = \gcd(m,\kappa)$ and $n \equiv \omega m+\kappa$, except when $\kappa = 0$. In this case we have on m-colored $GML_m^n\{\nu,\Theta\}$ body.*

Definition 2. Ribs of the $GML_m^n\{\nu,\Theta\}$ body are continuous closed lines, in which are situated only the vertices of the radial cross sections (plane figures) of this body. A single rib contains all A_i vertices present in the original prisms. These lines are always toroidal lines with characteristic ν for any m on n.

Remark 4. Part of the rib-line for any $GML_m^n\{\nu,\Theta\} \equiv GML_m^{\omega m+k}\{\nu,\Theta\}$ body, $k = 0,1,\ldots,m-1$, is part of the spatial closed curve and according to Definition 1 we may consider this as the section of GML (i.e. regular m-polygon) and this is a translation $A_0 \xrightarrow{n} A_{\omega m+k} \equiv V_k$, where V_k - is the corresponding vertex of the radial cross section of GML (i.e. regular m-polygon) and this is a translation after one full rotation.

• Full rib-line for any $GML_m^n\{\nu,\Theta\} \equiv GML_m^{\omega m+k}\{\nu,\Theta\}$ body, $k = 0,1,\ldots,m-1$, is a closed line and we may consider this as cycle of the corresponding element of group of permutations:

$$V_0 \xrightarrow{n} V_{\omega m+k} \equiv V_k \xrightarrow{n} V_{2\omega m+2k} \equiv V_{k_1} \xrightarrow{n} \cdots \xrightarrow{n} V_0 \qquad (3)$$

where the number of translations is $\frac{m}{\gcd(m,k)}$!

• According to Corollary 1: If $GML_m^n\{\nu,\Theta\}$ body is convex and has a j-colored surface i.e. body having j independent rib-lines, and then each rib-line is a $2\pi j$-periodic line and has $\frac{m}{\gcd(m,k)} \equiv \frac{m}{j}$ -translations. The corresponding permutation has j cycles with length m/j!

Definition 3. Side of the $GML_m^n\{\nu,\Theta\}$ body is a continuous closed surface, in which only the sides of the radial cross section (plane figures) of this body are situated.

Remark 5. According to Corollary 1: If $GML_m^n\{\nu,\Theta\}$ body is convex and has j-colored surfaces i.e. a body having j independent closed parts of the side-surface, then each side-surface has $\frac{m}{\gcd(m,k)} \equiv \frac{m}{j}$-translations (3), or this permutation has j cycles with length m/j!

When cutting the GML body with a single knife, one of three types of slit-surfaces appear, dependent on whether the cut is made from vertex to vertex ($VV_{0,k}$), from vertex to side ($VS_{0,k}$) or from side to side ($SS_{1,k}$).

Definition 4. 1. $S_{1,k} \equiv SS_{1,k}$-surface of the $GML_m^n\{\nu,\Theta\}$ body, $k = 2,3,\ldots,\left[\frac{m}{2}\right]+1$, is a slit-surface $GML_2^k\{\tilde{\nu},\tilde{\Theta}\}$ such that the ends of the straight line (radial cross section) are situated on the sides with the numbers 1 (or $A_0A_1 \equiv S_1$) and j (or $A_{k-1}A_k \equiv S_k$) where $k = 2,3,\cdots,\left[\frac{m}{2}\right]+1$ correspondingly of the plane figures (m-symmetric polygon) of the radial cross section of the $GML_m^n\{\nu,\Theta\}$ body (i.e. the cutting line has a beginning $C_k^1 \in S_1$ and an end $C_k^2 \in S_k$);

2. $VS_{0,k}$- surface of the $GML_m^n\{\nu,\Theta\}$ body, $k = 2,3,\ldots,\left[\frac{m-1}{2}\right]+1$, is a slit-surface $GML_2^k\{\tilde{\nu},\tilde{\Theta}\}$, whose radial cross section (straight line) is situated on the edges with the numbers j (where $j = 2,3$) and contains vertex number 0 of the radial cross section of the $GML_m^n\{\nu,\Theta\}$ body (i.e. the cutting line has a beginning $C_j^1 \equiv A_0 \equiv V_0$ and an end $C_k^1 \in S_k$);

3. $V_{0,k} \equiv VV_{0,k}$ - surface of the $GML_m^n\{\nu,\Theta\}$ body $k = 2,3,\ldots,\left[\frac{m}{2}\right]$ is a slit-surface $GML_2^k\{\tilde{\nu},\tilde{\Theta}\}$, whose radial cross sections (straight line) contain correspondingly vertices numbers 0 and k of the radial cross section of polygon, (i.e. the cutting line has a beginning $C_k^1 \equiv V_0$ and an end $C_k^2 \equiv V_k$). For all other values k, because of the symmetry of the figure, the results will be repeated.

Definition 5. These surfaces may be represented by (1) where function $p(x,z)$- is a d_m-knife, in other words, it is construction with d_m straight lines (the number d_m can only be a divisor of m) and has the form

$$\sin\left(\alpha+\frac{360}{m}i\right)x_i+\cos\left(\alpha+\frac{360}{m}i\right)z_i+\delta=0, i=0,1,\ldots,m-1,\ -\frac{360^\circ}{2m}\le\alpha\le\frac{360^\circ}{2m}. \quad (4)$$

The rotation parameter α and zoom parameter δ allow to connect d_m -knives to $VV_{0,k}, VS_{0,k}$ and $SS_{1,k}$ cuts. Figure 1 shows examples of the traces d_m -knives in a polygon of symmetry 16, i.e. correspondingly traces of $S_{1,5}$ in the cross section of $GML_{16}^n\{\nu, \Theta\}$.

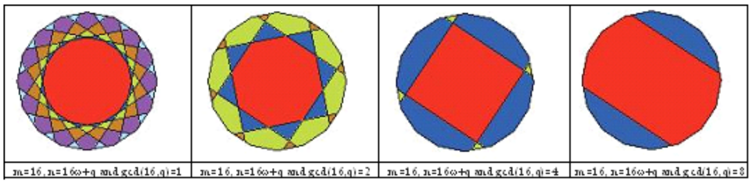

Figure 1:

Remark 6. The full cutting of $GML_m^n\{\nu, \Theta\}$ (see [3]) leaves the same trace in the cross section as the corresponding d_m-knife, but this same trace is achieved in only one rotation of 360° with the d_m –knife.

Remark 7. If $\delta = 0$ in (4) the d_m -knife cuts the centre of symmetry of polygon in $VV_{0,k}$ and $VS_{1,k}$ cuts. The necessary conditions for the Möbius phenomenon are 1. $\delta = 0$ in (4) and 2. m should be an even number.

REFERENCES

1. Tavkhelidze, I., Ricci, P.E. Rendiconti Accademia Nazionale dell Scienze detta dei XL Memorie di Matematica a Applicazioni. 124^0, XXX, 1, (2006), 191-212;

2. Tavkhelidze, I., Caratelli, D., Gielis, J., Ricci, P.E., Rogava, M. and Transirico, M. On a Geometric Model of Bodies with "Complex" Configuration and Some Movements. *Modeling in Mathematics - Chapter 10. Atlantis Transactions in Geometry 2*, DOI 10.2991/978-94-6239-261-$8_1$0 - Springer (2017), 129-158.

3. Tavkhelidze, I., Ricci, P.E. Some Properties of "Bulky" Links, Generated by Generalised Möbius-Listing's Bodies. *Modeling in Mathematics, Chapter 11 - Atlantis Transactions in Geometry 2*, DOI 10.2991/978-94-6239-261-$8_1$1 - Springer (2017), pp. 58-185.

Received 10.05.2018; revised 27.09.2018; accepted 20.10.2018.

Author(s) address(es):

Ilia Tavkhelidze
I. Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences
University str. 2, 0186 Tbilisi, Georgia
E-mail: ilia.tavkhelidze@tsu.ge

Johan Gielis
University of Antwerp
Department of Bio-Engineering Sciences, Belgium
E-mail: johan.gielis@uantwerpen.be