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SOLUTION OF ONE INTEGRAL EQUATION FROM MULTIVELOCITY TRANSPORT THEORY

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Abstract. The aim of this paper is to construct the continuous solution of the nonhomogeneous linear equation corresponding to the characteristic equation of the multivelocity transport theory in the isotropic case.

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Consider the following nonhomogeneous linear integral equation

$$(\nu-\mu)\tilde{\psi}_\nu(\mu,E)=\nu\int_{E_1}^{E_2}\int_{-1}^{+1}\tilde{\psi}_\nu(\mu',E')d\mu'dE'+f(\mu,E), \tag{1}$$

$$\mu\in(-1,+1),\quad E\in[E_1,E_2],$$

where the parameter ν is any point of the plane $f(\mu,E)$ is a continuous function satisfying H^* conditions [1] with respect to μ. Corresponding to this equation homogeneous equation

$$(\nu-\mu)\phi_\nu(\mu,E)=\nu\int_{E_1}^{E_2}\int_{-1}^{+1}\phi_\nu(\mu',E')d\mu'dE'$$

is the characteristic equation of the multi-velocity transport theory [2]. For this equation we can formulate the following results:

(a) There are two discrete eigenvalues $\pm\nu_0$,defined from the equation

$$\Lambda(\nu)\equiv 1-c\nu\ln\frac{1+1/\nu}{1-1/\nu}=0,$$

(here ($c=E_2-E_1<1$) and regular eigenfunctions

$$\phi_{0\pm}(\mu,E)=\frac{c\nu_0}{\nu_0\mp\mu};$$

(b) The continuum singular eigenfunctions

$$\phi_{\nu,(\zeta)}(\mu,E)=\frac{c\nu}{\nu-\mu}+\left(\delta(\zeta-E)-\int_{-1}^{+1}\frac{c\nu}{\nu-\mu'}d\mu'\right)\delta(\nu-\mu).$$

$$\nu\in(-1,+1),\quad\zeta\in[E_1,E_2].$$

The usefulness of these functions arises from the facts that the set of eigenfunctions $\{\phi_{\pm\nu_0}\} \cup \{\phi_{\nu,(\zeta)}\}$ is orthogonal and complete. This can be stated in the form of the following theorems (cf. [2,3]):

Theorem 1.
(a)

$$\int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_\nu(\mu,E)\phi_{\nu'}(\mu,E)d\mu dE = 0, \quad \nu \neq \nu';$$

(b)

$$\int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_{\nu,(\zeta)}(\mu,E)\tilde{\phi}_{\nu',(\zeta')}(\mu,E)d\mu dE = \delta(\nu-\nu')\delta(\zeta-\zeta'),$$

$$\nu,\nu' \in (-1,+1) \quad \zeta,\zeta' \in [E_1,E_2];$$

(c)

$$\int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_{\pm\nu_0}(\mu,E)\phi_{\nu,(\zeta)}(\mu,E)d\mu dE = 0, \quad \zeta \in [E_1,E_2].$$

Let

$$\tilde{\phi}_{\nu,(\zeta)}(\mu,E) = \phi_{\nu,(\zeta)}(\mu,E) + \frac{g(\nu)}{1-cg(\nu)}\int_{E_1}^{E_2}\phi_{\nu,(\zeta')}(\mu,E)d\zeta',$$

where

$$g(\nu) = -\pi^2\nu^2c + 2\int_{-1}^{+1}\frac{\nu}{\nu-\mu}d\mu - c\Big(\int_{-1}^{+1}\frac{\nu}{\nu-\mu}d\mu\Big)^2.$$

Theorem 2. *The arbitrary continuous function $\psi(\mu,E)$ defined in $-1<\mu<1$, $E_1 \leq E \leq E_2$ and satisfying H^* conditions with respect to μ, can be expressed in the form*

$$\psi(\mu,E) = a_{+\nu_0}\phi_{\nu_0}(\mu,E) + a_{-\nu_0}\phi_{-\nu_0}(\mu,E)$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1} u(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)d\nu d\zeta,$$

where

$$a_{\pm\nu_0} = \frac{1}{N_{\pm}\nu_0}\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{\pm\nu_0}(\mu,E)\psi(\mu,E)d\mu dE,$$

$$N_{\pm\nu_0} = \int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi^2_{\pm\nu_0}(\mu,E)d\mu dE$$

and

$$u(\nu,\zeta) = \int_{E_1}^{E_2}\int_{-1}^{+1}\mu\tilde{\phi}_{\nu,(\zeta)}(\mu,E)\psi(\mu,E)d\mu dE.$$

Let $\tilde{\psi}_\nu(\mu, E)$ be a solution of equation (1). From this theorem it can represented in the form

$$\tilde{\psi}_\nu(\mu, E) = b^{(\nu)}_{+\nu_0}\phi_{+\nu_0}(\mu, E) + b^{(\nu)}_{-\nu_0}\phi_{-\nu_0}(\mu, E)$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1} u^\nu(t, \zeta)\phi_{t,(\zeta)}(\mu, E)dtd\zeta,$$

Substituting this expression in equation (1) we obtain

$$f(\mu, E) = b^{(\nu)}_{+\nu_0}\frac{\nu_0}{\nu_0 - 1}\mu\phi_{+\nu_0}(\mu, E),$$

$$b^{(\nu)}_{-\nu_0}\frac{\nu_0}{\nu_0 + 1}\mu\phi_{-\nu_0}(\mu, E)$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1} R^{(\nu)}(t, \zeta)\Big(\frac{\nu}{t} - 1\Big)\mu\phi_{t,(\zeta)}(\mu, E)dtd\zeta.$$

Using Theorem 1, we obtain

$$b^{(\nu)}_{\pm\nu_0} = \frac{\pm\nu_0}{\nu \mp \nu_0}\frac{1}{N_{\pm\nu 0}}\int_{E_1}^{E_2}\int_{-1}^{+1} f(\mu, E)\phi_{\pm\nu_0}(\mu, E)d\mu dE,$$

$$R^{(\nu)}(t, \zeta) = \frac{t}{\nu - t}\int_{E_1}^{E_2}\int_{-1}^{+1} f(\mu, E)\tilde{\phi}_{\nu,(\zeta)}(\mu, E)f(\mu, E)d\mu dE.$$

Thus the following theorem is correct

Theorem 3. *If $\pm\nu_0\bar{\in}\{\nu_\pm\} \cup (-1, 1)$ equation (1) has only one continuous solution, satisfying conditions H^* with respect to μ and it can be represented in the form*

$$\tilde{\psi}_\nu(\mu, E) = \frac{+\nu_0}{\nu - \nu_{0+}}\frac{1}{N_{+\nu_0}}\int_{E_1}^{E_2}\int_{-1}^{+1} f(\mu', E')\phi_{+\nu_0}(\mu', E')d\mu' dE')\phi_{+\nu_0}(\mu, E)$$

$$+\frac{\nu_0}{\nu + \nu_0}\frac{1}{N_{-\nu_0}}\int_{E_1}^{E_2}\int_{-1}^{+1} f(\mu', E')\phi_{-\nu_0}(\mu', E')d\mu' dE')\phi_{-\nu_0}(\mu, E)$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1}\frac{t}{\nu - t}\int_{E_1}^{E_2}\int_{-1}^{+1}\tilde{\phi}_{t,(\zeta)}(\mu, E)f(\mu', E')d\mu' dE'\phi_{t,(\zeta)}(\mu, E)dtd\zeta,$$

$$\mu \in (-1, +1), \quad E \in [E_1, E_2].$$

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