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ON THE NUMBER OF REPRESENTATIONS OF INTEGERS BY THE QUADRATIC FORMS OF EIGHT VARIABLES

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Abstract. The spaces of spherical polynomials and the theta-series with respect to quadratic form are considered. The basis of these spaces for certain quadratic form is constructed and the number of representations of integers by the quadratic forms of eight variables is obtained.

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1 Introduction.

Let

$$Q(X) = Q(x_1, \dots, x_r) = \sum_{1 \le i \le j \le r} b_{ij} x_i x_j$$

be an integral positive definite quadratic form in an even number r of variables. That is, $b_{ij} \in \mathbb{Z}$ and $Q(X) > 0$ if $X \neq 0$. To $Q(X)$ we associate the even integral symmetric $r \times r$ matrix A defined by $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, where $i < j$. If $X = [x_1, \dots, x_r]'$ denotes a column vector where $'$ denotes the transposition, then we have $Q(X) = \frac{1}{2}X'AX$. Let A_{ij} denote the algebraic adjunct of the element a_{ij} in $D = \det A$ and a^*_{ij} the corresponding element of A^{-1}. $\Delta = (-1)^{\frac{r}{2}} D$ denotes the discriminant of the quadratic form $Q(X)$; $\delta = gcd\Big(\frac{1}{2}A_{ii}, A_{ij}\Big)$ $(i, j = 1, 2, ..., r)$, $N = \frac{D}{\delta}$ is the step of the quadratic form $Q(X)$; $\chi(d)$ is a character of the quadratic form $Q(X)$, i.e. if Δ is square, then $\chi(d) = 1$. A positive quadratic form of weight $\frac{r}{2}$, step N and character χ is called a quadratic form of type $\Big(\frac{r}{2}, N, \chi\Big)$.

A homogeneous polynomial $P(X) = P(x_1, \dots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1 \le i,j \le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]).

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$, and showed that

$$\dim P(\nu, Q) = \binom{\nu + r - 1}{r - 1} - \binom{\nu + r - 3}{r - 1}$$

and among homogenous quadratic polynomials in r variables

$$\varphi_{ij} = x_i x_j - \frac{A_{ij}}{rD} 2Q(X), \quad (i,j = 1, ..., r)$$

exactly $\frac{r(r+1)}{2} - 1$ ones are linearly independent and form the basis of the space of spherical polynomials of second order with respect to $Q(x)$.

It is known ([2, pp.808 and 855]) that, if $Q(x)$ is a quadratic form of type $(-k, N, \chi)$, $2|k$ and $P(X)$ is a spherical polynomial of order ν, then the generalized r-fold theta-series

$$\vartheta(\tau, P, Q) = \sum_{n \in \mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im} \tau > 0$$

is a parabolic form of type $\Big(-(k+\nu), N, \chi\Big)$.

Further $S_k(N, \chi)$ denotes the space of parabolic forms of type $(-k, N, \chi)$.

It is also known ([2, pp.874 and 817]) that, if $Q(x)$ is a quadratic form of type $(-k, q, 1)$, $2|k$, $k > 2$ and q is odd prime, then

$$\vartheta(\tau, Q(x)) = 1 + \sum_{n=1}^{\infty} r(n, Q) z^n \tag{1}$$

is the corresponding theta-series and

$$E(\tau, Q(x)) = 1 + \sum_{n=1}^{\infty} \Big(\alpha \sigma_{k-1}(n) z^n + \beta \sigma_{k-1}(n) z^{qn}\Big) \tag{2}$$

is the corresponding Eisenstein series for each positive quadratic form $Q(x)$, where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

In this paper we form the spherical polynomials of second order with respect to some quadratic form $Q(X)$, we form also the basis of the space of corresponding generalized theta-series and obtained the formulae for the number of representations of positive integers by quadratic form of eight variables.

2 Formulae for the number of representations of positive integers by quadratic form F. Consider the quadratic form

$$Q(X) = x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2 + x_1x_2 + x_1x_4 + x_2x_3 + x_2x_4 + 2x_3x_4,$$

it is of type $(-2, 13, 1)$ (see [3]). The spherical polynomials of second order with respect to quadratic form $Q(X)$ has the form:

$$\varphi_{11} = x_1^2 - \frac{4}{13} Q(X),$$

$$\varphi_{12} = x_1x_2 + \frac{1}{13}Q(X),$$

$$\varphi_{13} = x_1x_3 - \frac{1}{26}Q(X).$$

The corresponding generalized theta-series

$$\vartheta(\tau,\varphi_{11},Q) = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}\varphi_{11}\big)z^n = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}(x_1^2 - \frac{4}{13}Q(x))\big)z^n$$

$$= \frac{18}{13}z - \frac{22}{13}z^2 - \frac{18}{13}z^3 + ...,$$

$$\vartheta(\tau,\varphi_{12},Q) = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}\varphi_{12}\big)z^n = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}(x_1x_2 + \frac{1}{13}Q(x))\big)z^n$$

$$= \frac{2}{13}z - \frac{14}{13}z^2 - \frac{2}{13}z^3 + ...,$$

$$\vartheta(\tau,\varphi_{13},Q) = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}\varphi_{13}\big)z^n = \sum_{n=1}^{\infty}\big(\sum_{Q(x)=n}x_1x_3 - \frac{1}{26}Q(x)\big)z^n$$

$$= -\frac{1}{13}z - \frac{6}{13}z^2 + \frac{14}{13}z^3 + \cdots$$

are the parabolic forms of type $(-4, 13, 1)$ and they form the basis of the space $S_4(13, 1)$.

Consider now the quadratic form of eight variables

$$F = Q(x_1, x_2, x_3, x_4) + Q(x_5, x_6, x_7, x_8).$$

For this quadratic form the Eisenstein series has the form

$$E(\tau, F) = 1 + \frac{24}{17}\sum_{n=1}^{\infty}\Big(\sigma_3(n)z^n + 169\sigma_3(n)z^{13n}\Big) = 1 + \frac{24}{17}z + \frac{216}{17}z^2 + \frac{672}{17}z^3 + \cdots .$$

According to (1) theta-series for the quadratic form F has the form

$$\vartheta(\tau, F) = \vartheta^2(\tau, Q) = 1 + 4z + 16z^2 + 40z^3 + \cdots .$$

The difference $\vartheta(\tau, F) - E(\tau, F)$ is a parabolic form of type $(-4, 13, 1)$. We have constructed the basis of the space $S_4(13, 1)$, hence there are constants c_1, c_2 and c_3 such that

$$\vartheta(\tau, F) - E(\tau, F) = c_1\vartheta(\tau,\varphi_{11},Q) + c_2\vartheta(\tau,\varphi_{12},Q) + c_3\vartheta(\tau,\varphi_{13},Q).$$

Equating the coefficient of z, z^2 and z^3 on the both sides of this equality we obtain $c_1 = \frac{52}{17}$, $c_2 = -\frac{156}{17}$, $c_3 = \frac{52}{17}$.

Hence,

$$\vartheta(\tau,F)=E(\tau,F)+\frac{52}{17}\vartheta(\tau,\varphi_{11},Q)-\frac{156}{17}\vartheta(\tau,\varphi_{12},Q)+\frac{52}{17}\vartheta(\tau,\varphi_{13},Q).$$

Equating the coefficients of z^n on both sides of this identity we deduce the following result.

Theorem. *The number of representations of positive integers n by quadratic form F is given by*

$$r(n,F)=\frac{24}{17}\sigma_3^*(n)+\frac{91}{34}\Big(\sum_{Q(x)=n}x_1^2-\frac{4}{13}n\Big)$$
$$-\frac{247}{34}\Big(\sum_{Q(x)=n}x_1x_2-\frac{4}{13}n\Big)-\frac{26}{17}\Big(\sum_{Q(x)=n}x_1x_3\Big),$$

where

$$\sigma_3^*(n)=\begin{cases}\sigma_3(n),\ if(13,n)=1\\ \sigma_3(n)+169\sigma_3(\frac{n}{13}),\ if13|n\end{cases}.$$

REFERENCES

1. Gooding, F. Modular forms arising from spherical polynomials and positive definite quadratic forms. *J. Number Theory*, **9** (1977), 36-47.

2. Hecke, E. Mathematische Werke. *Vandenhoeck und Ruprecht,* Göttingen, 1970.

3. Shavgulidze, K. On the number of representacions of integers by the sums of quadratic forms $x_1^2+x_1x_2+3x_2^2$. *International Journal of Number Theory*, **5** (2009), 515-525.

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