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THE KERNEL ESTIMATION OF THE DENSITY AND ITS PRECISION IN THE CASE WITH CHAIN DEPENDENT ON THE SEQUENCES

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Abstract. In the paper stationary (in the narrow sense) two $-$ component sequence $\{\xi_i, X_i\}_{i\geq 1}$ is considered on the probability space (Ω, F, P). $\{X_i\}_{i\geq 1}$ is a sequence with chain dependence which is controlled by a finite regular Markov chain $\{\xi_i\}_{i\geq 1}$ with a set of states $\{b_1, b_2 \dots b_r\}$. X_i are observations over a random variable X and the conditional distributions $P_{X_1|\xi_1=b_i}$ $i=\overline{1,r}$ have unknown densities $f_i(x)$, $i=\overline{1,r}$ respectively. In certain conditions the precision $\bar{f}(x) = \sum_{i=1}^{r} p(\xi_1 = b_i)\ f_i(x)$ of density approximation is determined by core type it's estimation.

Keywords and phrases: Sequence with chainwise dependence, kernel estimate, Markov chain.

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1 Introduction. On the probability space (Ω, F, P), we consider the two-component, stationary in the narrow sense, sequence of random variables. The sequence $\{X_i\}_{i\geq 1}$ is a finite stationary homogeneous Markov chain.

$$\{\xi_i, X_i\}_{i\geq 1} \tag{1}$$

where the sequence $\{\xi_i\}_{i\geq 1}, (\xi_i : \Omega \to \Xi)$ is a finite stationary homogeneous regular Markov chain with a set of states $\Xi = \{b_1, b_2 \dots b_r\}$ the initial distribution $\pi = (\pi_1, \pi_2 \dots \pi_r), \pi_i = P(\xi_1 = \ b_i),\ \ \overline{1,r}$ and the matrix of transient probabilities P.

$\{X_i\}_{i\geq 1}$ is the sequence with chain dependence (S, C, D), ([1]) controlled by the sequence $\{\xi_i\}_{i\geq 1}$. For the fixed trajectory $\bar{\xi_{1n}} = (\xi_1\, \xi_2\,, \dots, \xi_n)$ the values $(X_1, X_2, \dots, X_n)$ become independent and for any natural numbers $i,\ r,\ n,\ j_1, j_2, \dots, j_r,\ (2 \leq r \leq n,$ $1 \leq i \leq n, 1 \leq j_1 < j_2 < \dots < j_r \leq n)$ we have the equalities

$$P_{(X_{j_1}, X_{j_2}, \dots, X_{j_r})|\xi_{1n}} = P_{(X_{j_1}|\xi_{j_1})} * P_{(X_{j_2}|\xi_{j_2})} * \dots * P_{(X_{j_r}|\xi_{j_r})}.$$

$$P_{X_i|\xi_{1n}} = P_{X_i|\xi_i}. \qquad 1 \leq i \leq n$$

where $P_{X|Y}$ is the angular distribution X with condition Y.

Assume that $X_i,\ i=\overline{1,r}$ are observations over some general population $L(x)$ and the distributions X_i have the unknown density depending on the values of ξ_i.

Assume further that the distributions $P_{X_i|\xi_i=b_m}$ $m=\overline{1,r}$ have the unknown densities $f_i(x)$, $i=\overline{1,r}$ respectively. Let us consider the empirical approximation of the density

$$f_1(x) + \pi_2 f_2(x) + \dots + \pi_r f_r(x) \tag{2}$$

by observations $X_1, X_2, \dots, X_n$.

Denote by W_s the set of functions $\varphi(x)$ having derivatives up to the s-th order $(s \geq 2)$ inclusive. $\varphi^{(s)}(x)$ is a continuous bounded function from the class $L_2(-\infty, \infty)$.

The function $k(x)$ is called a function of the class H_s ($s \geq 2$ is an even number) if

$$k(-x) = k(x), \quad \int_{-\infty}^{\infty} k(x)dx = 1, \quad sup|k(x)| \leq A < \infty, \quad \int_{-\infty}^{\infty} x^i k(x)dx = 0,$$

$$i = 1, 2, \ldots, s-1; \quad \int_{-\infty}^{\infty} x^s k(x)dx \neq 0; \quad \int_{-\infty}^{\infty} x^s |k(x)|dx < \infty. \tag{3}$$

When the distribution x_i does not depend on ξ_i i.e. $\{X_i\}_{i\geq 1}$ are independent, equally distributed random variables with density $g(x)$ the class of estimates generated by the kernel $k(x)$

$$\hat{g}_n(x,\ a_n) = \frac{a_n}{n} \sum_{j=1}^{n} k(a_n(x - X_j))$$

is considered as a density estimate in the works [2] and [3]. Here $\{a_n\}_{n\geq 1}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} a_n = \infty, \qquad a_n = o(n). \tag{4}$$

While the kernel $k(x)$ is some Lebesgue-integrable Borel function. G. Watson and M. Leadbetter [4] considered a more general kernel type estimate than $\hat{g}_n(x,\ a_n)$. G. Mania considered the results of [2] for the case of vectors $X_i \in R^k$, $(k > 1)$ [5]. E.A. Nadaraya [6] established the sufficient conditions for the convergence of $\hat{g}_n(x, a_n)$ to $g(x)$ in the uniform metric with probability 1. Along with Rosenblatt-Parzen type kernel estimates, projection type estimates are considered in [7] and [8] when the expansion of the kernel in terms of a system of orthonormalized functions is used.

Various numerical characteristics were considered as a measure of divergence between $\hat{g}_n(x, a_n)$ and $g(x)$ in [2], [7], [8], [9]. E. Nadaraya considered the average integral value of the error square in [6].

Let us formulate two results from [6].

Lemma 1. *(**E. Nadaraya** [6]) Let the independent, equally distributed random variables $X_1, X_2, \ldots,$ be observations over the general population $L(x)$ and have an unknown density $g(x)$. If $g(x) \in W_s \cap L_2(-\infty, \infty)$, $k(x) \in H_s \cap L_2(-\infty, \infty)$, $\{a_n\}_{n\geq 1}$ is defined by sequence (4) then the following equalities*

$$\int_{-\infty}^{\infty} D\hat{g}_n(x, a_n)dx = \frac{a_n}{n} \int_{-\infty}^{\infty} k^2(x)dx + o\Big(\frac{a_n}{n}\Big),$$

$$\int_{-\infty}^{\infty} [E\hat{g}_n(x,\ a_n) - g(x)]^2 dx = a_n^{-2s} \frac{\alpha^2}{(s!)^2} \int_{-\infty}^{\infty} [f^{(s)}(x)]^2 dx + o(a_n^{-2s})$$

are valid, where

$$\alpha = \int_{-\infty}^{\infty} x^s k(x)d(x).$$

In the present work, for sequence(1) we consider (as in[6]) by means of the empirical approximation the form

$$\hat{f}_n(x, a_n) = \frac{a_n}{n} \sum_{j=1}^{n} k(a_n(x - X_j)),$$

where $\{a_n\}_{n \geq 1}$ *is a sequence [4], while the divergence measure between them is*

$$u(a_n) = E \int_{-\infty}^{\infty} [\hat{f}_n(x, a_n) - \bar{f}(x)]^2 dx.$$

For the fixed trajectory $\bar{\xi_{1n}} = \xi_1, \xi_2, \ldots, \xi_n$ *we denote by* $\nu_n(i)$ $i = \overline{1, r}$ *the occurrences in which the first n members of the sequence* $\{\xi_i\}_{i \geq 1}$ *take the values* b_i $i = \overline{1, r}$ *respectfully.*

Lemma 2. *([10]) If* $\{\xi_i\}_{i \geq 1}$ *is a finite stationary homogeneous regular Markov chain with a set of states* $\{b_1, b_2 \ldots b_r\}$ *and the functions* $\nu_n^{(i)}$ $i = 1, 2, \ldots, r$ *show the number of steps (moment of time) made by the chain during the first n steps in the states* b_i, $i = \overline{1, r}$ *respectively then*

$$E \frac{\nu_n(i)}{n} = \pi_i, \quad D \frac{\nu_n(i)}{n} \leq \frac{C_i(\pi, P)}{n}, \tag{5}$$

where $\pi = \pi_1, \pi_2, \ldots, \pi_r$, $\pi_i = P(\xi_1 = i)$, $i = \overline{1, r}$ *while* P *is the matrix of transient probabilites.*

The following statement is valid.

Theorem. *Let the following conditions let be fulfilled for the sequence (1)* $f_i(x) \in W_s \cap L_2(-\infty, \infty), i = \overline{1, r}$, $k(x) \in H_s \cap L_2(-\infty, \infty)$ *and* $\{a_n\}_{n \geq 1}$ *be a sequence of form (4). Then for any natural n there holds the estimate*

$$u(a_n) \leq \Big(\sum_{i=1}^{r} M_i \Big)^2 + \frac{a_n}{n} \int_{-\infty}^{\infty} k^2(x) dx + \Big(\frac{1}{n} \sum_{i=1}^{r} C_i(\pi, P) + \sum_{i=1}^{r} \pi_i^2 \Big) o\Big(\frac{a_n}{n} \Big),$$

where

$$M_i = T_i^{1/2} + \Big(\frac{C_i(\pi, P)}{n} \int_{-\infty}^{\infty} f_i^2(x) dx \Big)^{1/2}$$

and

$$T_i = \Big({a_n}^{-2s} \frac{\alpha^2}{(s!)^2} \int_{-\infty}^{\infty} [f^{(s)}(x)]^2 dx + o({a_n}^{-2s}) \Big) \Big(\frac{1}{n} (C_i(\pi, P) \Big) + \pi_i^2 \Big), \qquad \overline{1, r}.$$

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