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NUMERICAL CALCULATIONS OF THE J. BALL NONLINEAR DYNAMIC BEAM ⋆

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Abstract. An initial-boundary value problem is posed for the J. Ball integro-differential equation, which describes the dynamic state of a beam. The solution is approximated with respect to a spatial and a time variables by the Galerkin method and stabile symmetrical difference scheme, which requires carrying out of iteration process. The algorithm has been approved on tests and the results of recounts are represented in tables.

Keywords and phrases: Nonlinear dynamic beam equation, J. Ball equation, Galerkin method, implicit symmetric difference scheme, Jacobi iterative method, numerical realization.

AMS subject classification (2010): 65M60, 65M06, 65Q10, 65M15.

1 Statement of the problem.

Let us consider the nonlinear equation

$$\begin{gathered} u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t) \\ -\left(\beta+\rho\int\limits_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)-\sigma\left(\int\limits_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right) \\ \times u_{xx}(x,t)=f(x,t), \quad 0<x<L, \quad 0<t\le T, \end{gathered} \tag{1}$$

with the initial boundary conditions

$$\begin{gathered} u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \\ u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0. \end{gathered} \tag{2}$$

Here $\alpha,\gamma,\rho,\sigma,\beta$ and δ are given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$ are given functions such that $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$. It will be assumed that the inequality $|\delta|<\gamma\left(\frac{\pi}{L}\right)^4$ is fulfilled when $\delta<0$ and $\alpha\left(\frac{\pi}{L}\right)^2>|\beta|$ holds when $\beta<0$. Equation (1) is obtained by J.Ball [1] using the Timoshenko theory describes the vibration of a beam. The right-hand side $f(x,t)\in L_2((0,L)\times(0,T))$. We suppose that there exits a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ of problem (1), (2).

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2 Algorithm.

2.1 Galerkin method. We write an approximate solution of problem (1), (2) in the form $u_n(x,t) = \sum_{i=1}^{n} u_{ni}(t) \sin \frac{i\pi x}{L}$, where the coefficients $u_{ni}(t)$ will be found by the Galerkin method from the system of ordinary differential equations

$$\begin{gathered} u''_{ni}(t) + \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right) u'_{ni}(t) + \left\{\alpha\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2 \right. \\ \times \left.\left[\beta + \rho\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2(t) + \sigma\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}(t)\, u'_{nj}(t)\right]\right\} \\ \times u_{ni}(t) = f_{ni}(t), \qquad i = 1,2,\ldots,n, \qquad 0 < t \le T, \end{gathered} \tag{3}$$

with the initial conditions

$$u_{ni}(0) = a_i^0, \quad u'_{ni}(0) = a_i^1, \quad i = 1,2,\ldots,n, \tag{4}$$

where

$$a_i^p = \frac{2}{L}\int_0^L u^p(x) \sin \frac{i\pi}{L} x dx, \qquad p = 0,1,$$

$$f_{ni}(t) = \frac{2}{L}\int_0^L f(x,t) \sin \frac{i\pi}{L} x dx, \qquad i = 1,2,\ldots,n.$$

The error of a Galerkin method is estimated in [2].

2.2 Difference scheme. To solve problem (3), (4) we apply the difference method. On the time interval [0,T] we introduce a net with step $\tau = \frac{T}{M}$ and nodes $t_m = m\tau$, $m = 0,1,2,\ldots,M$.

On the m-th layer, i.e. for $t = t_m$, the approximate value of $u_{ni}(t_m)$ is denoted by u_{ni}^m and the right side $f_{ni}(t_m) = f_{ni}^m$.

We use an implicit symmetric difference scheme (see [3]) where the homogeneous type equation is considered. Applying the symmetric difference scheme for the system of equations (3) on the right side will get the following values $f_{ni}(t_m)$, $m = 0,1,2,\ldots,M-1$.

The error of a difference scheme is estimated in [3].

2.3 Iterative method. The system obtained by the discretization will be solved layer-by-layer. Assuming that the solution has already been obtained on the $(m-1)$-th and m-th layer to find it on the $(m+1)$-th layer we use the Jacobi iterative method. For the sake of simplicity, the error of the final iterative approximation on the $(m-1)$-th and m-th layers will be neglected.

The convergence of the Jacobi iterative method is estimated in [4].

3 The numerical realization. For the approximate solution of initial-boundary value problem (1), (2) the several programs in "Maple" are composed and many numerical experiments are carried out. The obtained results are good enough.

The algorithm is approved in the following task on the test and their illustrations in the article are given.

We consider a special case, where $L = 1,\ T = 1,\ n = 5,\ M = 20,\ \tau = \dfrac{T}{M} = 0.05,\ H = 20,\ h = \dfrac{L}{H} = 0.05,\ k_0 = 10,\ \alpha = 1,\ \gamma = 1,\ \rho = 1,\ \sigma = 1,\ \beta = -1,\ \delta = -1.$

Task. Exact solution $u(x,t) = \sin\dfrac{\pi x}{L} + t^3 \sin\dfrac{2\pi x}{L}$, the initial functions $u^0(x) = \sin\dfrac{\pi x}{L}$, $u^1(x) = 0$, the right-hand side

$$f(x,t) = \left(6t + \delta\, 3t^2 + \gamma\left(\frac{2\pi}{L}\right)^4 3t^2\right)\sin\frac{2\pi x}{L} + \alpha\left(\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} + \left(\frac{2\pi}{L}\right)^4 t^3 \sin\frac{2\pi x}{L}\right)$$
$$+\left(\beta + \rho\left(\frac{\pi}{L}\right)^2 \frac{L}{2}\left(1+4t^6\right) + \sigma\left(\frac{2\pi}{L}\right)^2 \frac{L}{2} 3t^5\right)\cdot\left(\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L} + \left(\frac{2\pi}{L}\right)^2 t^3 \sin\frac{2\pi x}{L}\right).$$

$t \setminus x$	0	h	5h	9h	10h	11h	15h	19h	20h
t=0	0	0.1564	0.7071	0.9877	1.0	0.9877	0.7071	0.1564	0.0
t=τ	0	0.1565	0.7072	0.9877	1.0	0.9876	0.7070	0.1564	0.0
t=5τ	0	0.1613	0.7227	0.9925	1.0	0.9826	0.6915	0.1516	0.0
t=9τ	0	0.1846	0.7982	1.0158	1.0	0.9595	0.6160	0.1283	0.0
t=10τ	0	0.1951	0.8321	1.0263	1.0	0.9491	0.5821	0.1178	0.0
t=11τ	0	0.2078	0.8735	1.0391	1.0	0.9363	0.5407	0.1050	0.0
t=15τ	0	0.2868	1.1290	1.1181	1.0	0.8573	0.2852	0.2607	0.0
t=19τ	0	0.4214	1.5645	1.2526	1.0	0.7227	-0.1503	-0.1085	0.0
t=20τ	0	0.4655	1.7071	1.2967	1.0	0.6787	-0.2929	-0.1526	0.0

Table 1. Exact solution

$t \setminus x$	0	h	5h	9h	10h	11h	15h	19h	20h
t=0	0	0.1564	0.7071	0.9877	1.0	0.9877	0.7071	0.1564	0.0
t=τ	0	0.1564	0.7071	0.9877	1.0	0.9877	0.7071	0.1564	0.0
t=5τ	0	0.1610	0.7220	0.9922	0.9999	0.9830	0.6921	0.1518	0.0
t=9τ	0	0.1838	0.7950	1.0125	0.9970	0.9569	0.5791	0.1281	0.0
t=10τ	0	0.1938	0.8269	1.0203	0.9942	0.9437	0.5342	0.1172	0.0
t=11τ	0	0.2060	0.8654	1.0287	0.9897	0.9263	0.4787	0.1036	0.0
t=15τ	0	0.2791	1.0903	1.0531	0.9310	0.7861	0.2264	0.1216	0.0
t=19τ	0	0.3996	1.4360	0.9907	0.7111	0.41140	-0.4303	-0.1771	0.0
t=20τ	0	0.4396	1.5480	0.9589	0.6247	0.2751	-0.6646	-0.2442	0.0

Table 2. Approximate solution

4 Conclusion.

From numerical experiments it is clear:

1. Convergence of the Jacobi iterative process;

2. Questions of accuracy of the algorithms with respect to time and spatial variables both in the case of the Galerkin method and when using the symmetric difference scheme.

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