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STATISTICAL SUMS AND ZETA-FUNCTIONS

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Abstract. Fermion factorization of the bosonic statistical sum and finite approximation of the zeta-function considered.

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We say that we find **New Physics** (NP) when either we find a phenomenon which is forbidden by standard model (of elementary particles) (SM) in principal - this is the qualitative level of NP - or we find a significant deviation between precision calculations in SM of an observable quantity and a corresponding experimental value.

In the Universe, matter has manly two geometric structures, homogeneous, and hierarchical. The homogeneous structures are naturally described by real numbers with an infinite number of digits in the fractional part and usual archimedean metrics. The hierarchical structures are described with p-adic numbers with an infinite number of digits in the integer part and non-archimedean metrics [1]. A discrete, finite, regularized, version of the homogenous structures are homogeneous lattices with constant steps and distance rising as arithmetic progression. The discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

Qvelementary particles. Let us consider the following formula

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)..., \ |x| < 1. \tag{1}$$

which can be proved as

$$p_k \equiv (1+x)(1+x^2)(1+x^4)...(1+x^{2^k}) = \frac{1-x^{2^{(k+1)}}}{1-x}, \ \lim_{k\to\infty} p_k = 1/(1-x). \tag{2}$$

The formula (1) remind us the boson and fermion statistical sums

$$Z_b(\omega) = \frac{\sqrt{x}}{1-x}, \ Z_f(\omega) = \frac{1+x}{\sqrt{x}}, \ x = \exp(-\beta\omega) \tag{3}$$

and can be transformed in the following relation

$$Z_b(\omega) = Z_f(\omega)Z_f(2\omega)Z_f(4\omega)... \tag{4}$$

Indeed,

$$
\begin{aligned}
&Z_b(\omega) = \frac{\sqrt{x}}{1-x} = x^a Z_f(\omega) Z_f(2\omega) Z_f(4\omega)...,\\
&a = 1 + (1 + 2 + 2^2 + ...) = 1 + \frac{1}{1-2} = 0,\ |2|_2 = 1/2 < 1.
\end{aligned}
\tag{5}
$$

By the way we have an extra bonus! We see that the fermi content of the boson wears the p-adic sense. The prime $p = 2$, in this case. Also, the vacuum energy of the oscillators wear p-adic sense.

What about other primes p? For the finite fields,

$$
\begin{aligned}
&z_n(p) = \exp(2\pi i n/p),\ n = 0, 1, ..., p-1,\ \sum_n z_n = 0,\\
&Z_p(\beta) = \sum_{n=1}^{p-1} \exp\left(-\beta E_n/\hbar\right),\ E_n = 2\pi\hbar(n + a),\\
&Z_p(-i/p) = 0,\ p = 2, 3, 5, ...13...29...137...
\end{aligned}
\tag{6}
$$

In polynomial approximation of a function $f(x) \simeq P_N(x) = a_0 + a_1 x + ... + a_N x^N$,

$$
\begin{aligned}
&a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_N x_0^N = f(x_0) = f_0,\\
&a_0 + a_1 x_1 + a_2 x_1^2 + ... + a_N x_1^N = f(x_1) = f_1,\\
&...,\\
&a_0 + a_1 x_1 + a_2 x_N^2 + ... + a_N x_N^N = f(x_N) = f_N,
\end{aligned}
\tag{7}
$$

the coefficients $a_n,\ n = 0, 1, ..., N$ are defined as solutions of the linear system of equations

$$
\begin{aligned}
&VA = F,\ A^T = (a_0, a_1, ..., a_N),\ F^T = (f_0, f_1, ..., f_N),\\
&V = \begin{pmatrix} 1 & x_0 & x_0^2 & ... & x_0^N \\ 1 & x_1 & x_1^2 & ... & x_1^N \\ 1 & x_2 & x_2^2 & ... & x_2^N \\ . & . & . & . & . \\ 1 & x_N & x_N^2 & ... & x_N^N \end{pmatrix}
\end{aligned}
\tag{8}
$$

Determinant of the Vandermonde matrix $detV = \Delta_N = \prod_{N \geq m > n \geq 0}(x_m - x_n),\ (\Delta_0 = 1,$ by definition). Indeed,

$$
\begin{aligned}
&\Delta_1 = x_1 - x_0,\ \Delta_2 = det\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = det\begin{pmatrix} 1 & x_0 & x_0^2 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 \\ 0 & x_2 - x_0 & x_2^2 - x_0^2 \end{pmatrix}\\
&= (x_1 - x_0)(x_2 - x_0) det\begin{pmatrix} 1 & x_1 + x_0 \\ 1 & x_2 + x_0 \end{pmatrix} = (x_2 - x_1)(x_2 - x_0)(x_1 - x_0),\\
&\Delta_N = (x_N - x_{N-1})...(x_N - x_0)\Delta_{N-1} = \prod_{1 \leq n \leq N} Z_n,\\
&Z_n = (x_n - x_{n-1})...(x_n - x_0)
\end{aligned}
\tag{9}
$$

There are two exceptional (simplest) case for discrete values of x: when $x_n = p^n,\ n = 0,1,2,...,N$, and $x_n = x_0 + nh,\ n = 0,1,2,...,N$.

In the first, geometric progression, case

$$\begin{aligned}
&Z_n = (p^n - p^{n-1})(p^{n-1} - p^{n-2})...(p^n - 1)\\
&= p^{(1+2+...+n-1)}(p-1)^n \frac{p^n-1}{p-1}\frac{p^{n-1}-1}{p-1}...\frac{p-1}{p-1}\\
&= p^{n(n-1)/2}(p-1)^n[n]_p!,\ [n]_p = \frac{p^n-1}{p-1},\ \Delta_1 = Z_1 = (p-1),\\
&\Delta_2 = (p^2-p)(p^2-1)(p-1) = p(p-1)^3(p+1)\\
&= Z_2 Z_1 = p(p-1)^2(p+1)(p-1),\\
&\Delta_N = \prod_{1\le n\le N} Z_n = p^a(p-1)^b \prod_{1\le n\le N}[n]_p!,
\end{aligned} \tag{10}$$

$$\begin{aligned}
a &= \frac{1}{2}\sum_0^N n(n-1) = \frac{1}{2}(\sum_0^N x^n)^{(2)}|_{x=1} = \frac{1}{2}\frac{(1+\varepsilon)^{N+1}-1}{\epsilon}|_{\varepsilon=0}\\
&= \frac{1}{2}(N+1+\frac{(N+1)N}{2}\varepsilon + \frac{(N+1)N(N-1)}{3!}\varepsilon^2 + ...)^{(2)}|_{\varepsilon=0}\\
&= \frac{(N+1)N(N-1)}{6},\\
b &= \sum_0^N n = N(N+1)/2,\ \Delta_2 = p(p-1)^3(p+1),\ a=1,\ b=3.
\end{aligned} \tag{11}$$

For $p \gg 1$,

$$\begin{aligned}
&[n]_p \simeq p^{n-1},\ [n]_p! \simeq p^{n(n-1)/2},\\
&\Delta_N \simeq p^{2a+b} = p^c,\ c = \frac{N(N+1/2)(N+1)}{3} = \sum_1^N n^2,\\
&\Delta_1 \simeq p,\ \Delta_2 \simeq p^5.
\end{aligned} \tag{12}$$

For $p \ll 1, \Delta_N \simeq (-1)^b p^a,\ a = N(N^2-1)/6,\ b = N(N+1)/2,\ [n]_p \simeq 1,\ \Delta_1 \simeq -1,\ \Delta_2 \simeq -p$. Having expression for Δ_N in p, it is easy to obtain corresponding expression in arithmetic progression case by putting $p = 1+h:\ \Delta_N(h) = h^b \prod_1^N n!,\ b = N(N+1)/2, \Delta_2 = 2h^3$. We obtain the same result by direct calculation: $Z_n = h \times 2h \times ... \times nh = h^n n!,\ \Delta_N(h) = \prod Z_n$.

The Riemann zeta function $\zeta(s)$ is defined for complex $s = \sigma + it$. All complex zeros, $s = \alpha + i\beta$, lie in the critical stripe $0 < \sigma < 1$, symmetrically with respect to the real axe and critical line $\sigma = 1/2$. So it is enough to investigate zeros with $\alpha \le 1/2$ and $\beta > 0$. These zeros are of three types, with small, intermediate and big ordinates. The **Riemann hypothesis** states that the (non-trivial) complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$. The Riemann hypothesis (RH) is a central problem in Pure Mathematics due to its connection with Number theory and other branches of Mathematics and Physics.

Let us consider the following finite approximation of the Riemann zeta function

$$\zeta_N(s) = \sum_{n=1}^{N} n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-t} - e^{-(N+1)t}}{1 - e^{-t}} = \zeta(s) - \Delta_N(s), \ Re\ s > 1$$
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t - 1}, \ \Delta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1} e^{-Nt}}{e^t - 1} \tag{13}$$

Another formula, which can be used on the critical line, is

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n \geq 1} (-1)^{n+1} n^{-s} = \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t + 1}, \ Re\ s > 0. \tag{14}$$

Corresponding finite approximation of the Riemann zeta function is

$$\zeta_N(s) = (1 - 2^{1-s})^{-1} \sum_{n=1}^{N} (-1)^{n-1} n^{-s}$$
$$= \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}(1 - (-e^{-t})^N) dt}{e^t + 1} = \zeta(s) - \Delta_N(s),$$
$$\Delta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}(-e^{-t})^N)}{e^t + 1} \sim \pm N^{-s} \tag{15}$$

at a (nontrivial) zero of the zeta function, s_0, $\zeta_N(s_0) = -\Delta_N(s_0)$. In the integral form, dependence on N is analytic and we can consider any complex valued N.

It is interesting to see dependence (evolution) of zeros on N. For the simplest nontrivial integer $N = 2$, $\zeta_2(s) = (1 - 2^{1-s})^{-1}(1 - 2^{-s})$, we have zeros at $s_n = 2\pi i n / \ln 2$, $n = 0, \pm 1, \pm 2, ...,$ $2\pi / \ln 2 = 9.06$. In the interval $Im s_n \in (0, 100)$ we have 10 nontrivial zeros. The first nontrivial zero of the zeta function, by Mathematica, is: $s_1 = 1/2 + i14.1347$. The last zero in the interval $Im s_n \in (0, 100)$ is: $s_{29} = 1/2 + i98.8312$.

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