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ON THE GENERALIZED SOLUTION OF SOME NONLOCAL BOUNDARY PROBLEMS

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Abstract. For the simplest nonlocal boundary problem, the concept of generalized solution is introduced. The theorem on the continuous dependence of the solution from the right side of the equation and the note on the approximation of the solution is given.

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Let us consider the nonlocal boundary problem [1]: find the function, satisfying the problem for the given right side $f(x, y)$:

$$-\triangle \upsilon(x,y) + \lambda \upsilon(x,y) = f(x,y), \quad (x,y) \in G,$$

$$\upsilon(x,y)|_{\Gamma} = 0, \tag{1}$$

$$\upsilon(x,y)|_{\Gamma-\xi} = \upsilon(x,y)|_{\Gamma_0},$$

where $\lambda \geq 0, \Gamma = \partial G \backslash \Gamma_0$ and $\xi \in]0, a[$. By ∂G boundary of the rectangle $G = \{(x,y)| -a < x < 0, 0 < y < b\}$ is denoted and Γ_t is the intersection of the line $x = t \quad (-a \leq t \leq 0)$ with the $\bar{G} = G \bigcup \partial G$.

Many scientists have been investigating nonlocal boundary value problems for ordinary differential equations and partial differential elliptic equations (see, for example, [1] - [7] and references therein).

It is known that if $f(x,y) \in C(\bar{G})$ then problem (1) has the unique solution $u(x,y) \in C^{(2)}(G) \bigcap C(\bar{G})$ (classical solution) [1]. The purpose of this work is to generalize the concept of the classical solution of the problem for the case $f(x,y) \in L_2(G)$.

Let us recall some notations and facts from [2].

Let us denote by $D(\bar{G})$ the lineal of all the real functions $\upsilon(x,y)$ satisfying the following conditions:

1. $\upsilon(x,y)$ is defined almost everywhere on $\bar{G}$, and the boundary value $\upsilon(0,y)$ is defined almost everywhere on Γ_0 ;

2. $\upsilon(x,y) \in L_2(G), \quad \upsilon(0,y) \in L_2(0,b)$.

On the lineal $D(\bar{G})$ let us define the operator of symmetrical extension as follows

$$\tau\upsilon(x,y) = \begin{cases} \upsilon(x,y), & (x,y) \in \bar{G}, \\ -\upsilon(-x,y) + 2\upsilon(0,y), & (x,y) \in \bar{Q}, \end{cases}$$

where $Q = \{(x, y)|0 < x < \xi, \quad 0 < y < b\}$. Below we will use the notation $\tau\upsilon(x, y) = \tilde{\upsilon}(x, y)$.

For two arbitrary functions $g(x, y)$ and $h(x, y)$ from the lineal $D(\bar{G})$, let us define the following scalar product

$$[g, h] = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \tilde{g}(s, y)\tilde{h}(s, y)dsdxdy.$$

After the introduction of above scalar product the lineal $D(\bar{G})$ becomes the pre-Hilbert space. Let us denote it by $H(\bar{G})$ and the corresponding norm by $\|\cdot\|_H$, which is determined from the scalar product

$$\|\upsilon\|_H = [\upsilon, \upsilon]^{\frac{1}{2}}. \tag{2}$$

The norm (2), defined on the lineal $H(\bar{G})$, is equivalent to the norm, defined by the following formula

$$\|\upsilon\|^2 = \|\upsilon(x, y)\|^2_{L_2(G)} + \|\upsilon(0, y)\|^2_{L_2(0,b)}.$$

So, $H(\bar{G})$, is the Hilbert space.

Assume that the domain of definition of the operator $A = -\Delta + \lambda I$ is the lineal $D_A(\bar{G})$ of the functions from space $H(\bar{G})$, each $\upsilon(x, y)$ element of which satisfies the following conditions:

1. $\upsilon(x, y) \in C^{(2)}(\bar{G}), \quad \frac{\partial^2 \upsilon(-\xi, y)}{\partial x^2} = 0, \quad \frac{\partial^k \upsilon(0, y)}{\partial x^k} = 0, \quad \forall y = [0, b], \quad k = 1, 2;$
2. $\upsilon(x, y)|_\Gamma = 0, \quad \upsilon(x, y)|_{\Gamma-\xi} = \upsilon(x, y)|_{\Gamma_0}.$

The lineal $D_A(\bar{G})$ is dense in the space $H(\bar{G})$ and $A = -\Delta + \lambda I$ is positively defined operator on the lineal $D_A(\bar{G})$. One can follow to the standard way of completion of lineal $D_A(\bar{G})$ to the energetic space [8]. Let us introduce the new scalar product on $D_A(\bar{G})$:

$$[g, h]_A = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \left(\frac{\partial \tilde{g}(s, y)}{\partial s} \frac{\partial \tilde{h}(s, y)}{\partial s} + \frac{\partial \tilde{g}(s, y)}{\partial y} \frac{\partial \tilde{h}(s, y)}{\partial y} + \lambda \tilde{g}(s, y)\tilde{h}(s, y) \right) dsdxdy.$$

For the corresponding norm we use the notation $\|\cdot\|_A$. After introducing above scalar product, the lineal $D_A(\bar{G})$ becomes the pre-Hilbert space which we denote by $S_A(\bar{G})$. By $H_A(\bar{G})$ we denote the Hilbert space obtained after completion of $S_A(\bar{G})$ by the norm $\|\cdot\|_A$. The norm $|||\cdot|||$ in this space defined by the formula

$$|||\upsilon|||^2 = \|\upsilon\|^2_{W_2^{(1)}(G)} + \|\upsilon(0, y)\|^2_{W_2^{(1)}(0,b)}$$

and $\|\cdot\|_A$ are equivalent norms. Thus, any function $\upsilon(x, y)$ of the space $H_A(\bar{G})$ is the element of the space $W_2^{(1)}(G)$ and its $\upsilon|_{\Gamma-\xi}$ and $\upsilon|_{\Gamma_0}$ traces are the same element of the space $W_2^{(1)}(0, b)$.

Let $f_0(y) \in L_2(0, b)$ and $f(x, y) \in L_2(G)$, then for the function $\bar{f}(x, y) = (f(x, y), f_0(y)) \in H(\bar{G})$ the quadratic functional

$$F_{f_0(y)}(\upsilon) = [\upsilon, \upsilon]_A - 2[\bar{f}, \upsilon] \tag{3}$$

has the unique function $u_{f_0(y)}(\upsilon) \in H_A(\bar{G})$, which minimizes the functional (3) and for every $\upsilon(x, y) \in H_A(\bar{G})$ satisfies the identity

$$[u_{f_0(y)}, \upsilon]_A = [\bar{f}, \upsilon]. \tag{4}$$

Note that in minimization of the functional (3) the function $f(x, y)$ is fixed from problem (1), but the function $f_0(y)$ does not participate in the statement of problem (1). It changes in the space $L_2(0, b)$ and to every concrete value $f_0(y)$ corresponds only one minimizing $u_{f_0(y)}(x, y) \in H_A(\bar{G})$ function. If $f(x, y) \in C(\bar{G})$ and $f_0(y)$ is that the corresponding minimizing function $u_{f_0(y)}(x, y)$ is smooth enough, then from (4) we get

$$Au_{f_0(y)}(x, y) = f(x, y)$$

and $u_{f_0(y)}(x, y)$ is the classical solution of problem (1): $u_{f_0(y)}(x, y) = u(x, y)$.

When $f(x, y) \in C(\bar{G})$, then for problem (1) the certain variational equivalent is the following statement.

Theorem 1. *The function $u_{f_0(y)}(x, y) \in H_A(\bar{G})$ which minimizes the functional (3), is a solution of problem (1) if and only if the following condition is fulfilled [3]*

$$-\frac{d^2 u_{f_0(y)}(0, y)}{dy^2} + \lambda u_{f_0(y)}(0, y) = f_0(y), \quad y \in]0, b[.$$

Therefore, when $f_0(y) = -\frac{d^2 u(0,y)}{dy^2} + \lambda u(0, y)$, then $u(x, y)$ will be a minimizing function of the functional (3). The uniqueness of the selected $f_0(y)$ function is derived from the uniqueness of the classical $u(x, y)$ solution.

The main content of the presented work is expressed in the following statement of the continuous dependence of the solution of problem (1) on the right side.

Theorem 2. *For problem (1) there exists a constant $C > 0$, such that for any $f(x, y) \in C(\bar{G})$ function and its corresponding solution $u(x, y)$ satisfies the inequality*

$$\|u(x, y)\|_{W_2^{(1)}(G)} \leq C\|f(x, y)\|_{L_2(G)}. \tag{5}$$

Let now $f(x, y) = \varphi(x, y), \varphi(x, y) \in L_2(G)$ and let $\{f_n(x, y)\}$ be the sequence of continuous functions on the domain $\bar{G}$ converging to the function $f(x, y)$. It is easy to see that for all of such sequences $f_n(x, y)$ the classical solutions converge to the function $\bar{u}(x, y)$ of the space $W_2^{(1)}(G)$ and

$$\|\bar{u}(x, y)\|_{W_2^{(1)}(G)} \leq C\|\varphi(x, y)\|_{L_2(G)}. \tag{6}$$

It is clear that, if $\varphi(x, y) \in C(\bar{G})$, then $\bar{u}(x, y)$ represents the classical solution of problem (1).

It is natural to introduce the following definition.

Definition. The generalized solution of problem (1) for the right side $\varphi(x, y) \in L_2(G)$, (or simply generalized solution) is said to be the function $\bar{u}(x, y) \in W_2^{(1)}(G)$, which is the limit in $W_2^{(1)}(G)$ of the classical solutions $\{u_n(x,y)\}$ of problem (1) with continuous on $\bar{G}$ right side functions $\{f_n(x,y)\}$ converging to the function $\varphi(x,y) \in L_2(G)$.

Inequality (6) expresses the continuous dependence of the generalized solution of problem (1) on the right side $\varphi(x,y) \in L_2(G)$.

Notice that if $f(x,y) \in A(D_A(\bar{G}))$ and $\bar{f}(x,y) = (f(x,y), f(0,y))$ then minimizing function of the functional $F_{f(0,y)}(\upsilon)$ is the solution of the problem (1), in this case $u_{f(0,y)}(x,y) = u(x,y) \in D_A(\bar{G})$. Thus, if $\varphi(x,y) \in L_2(G), \varepsilon > 0$ and $\bar{u}(x,y)$ is the generalized solution of problem (1) with the right side $\varphi(x,y)$ and function $f(x,y) \in A(D_A(\bar{G}))$ approximates $\varphi(x,y)$ function with accuracy ε

$$\|\varphi(x,y) - f(x,y)\|_{L_2(G)} < \varepsilon,$$

then using (6) we have

$$\|\bar{u}(x,y) - u(x,y)\|_{W_2^{(1)}(G)} < \varepsilon C.$$

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