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THE BETA FUNCTION AND THE WEAK LIMITS RELATED WITH IT *

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Abstract. We studied the weak limits connected with the $B(\alpha, \beta)$ beta function, for the particular cases of the parameters. It turned out, that in some area of divergence of the beta function the weak limits related to it can be expressed by the singular generalized functions, for instance, with the Dirac and Sokhotsky functions.

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1 Introduction. It's known that the Euler Beta function $B(\alpha, \beta)$ is an analytic function in the domain of its definition and one can express it as follows (see, e. g., [1], p. 246):

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \ \mathrm{Re}\,(\alpha) > 0, \ \mathrm{Re}\,(\beta) > 0, \tag{1}$$

where $\Gamma(z)$ is the Euler Gamma function (see, e. g., [1], p. 238):

$$\Gamma(z) = \int_0^1 dt t^{z-1} \exp(-t), \ \ \mathrm{Re} z > 0.$$

We have defined the Beta function of the imaginary parameters $B(i\tau, -i\tau)$ as a weak limit of analytic functions, and have shown [2]:

$$B(i\tau, -i\tau) = \lim_{\varepsilon \to 0+} B(\varepsilon + i\tau, \varepsilon - i\tau) = 2\pi\delta(\tau). \tag{2}$$

We also studied the limiting $\varepsilon \to 0+$ behaviour of the following sequences of the analytic functions $B(\varepsilon + i\tau + n, \varepsilon - i\tau)$. The new results obtained will be suggested below.

2 Content. Let calculate the weak limit:

$$\lim_{\varepsilon \to 0+} B(\varepsilon + i\tau + n, \varepsilon - i\tau), \quad n = 1, \ 2, \ . \ \ . \ \ .$$

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According to formula (1), one can write:

$$B\left(\varepsilon+i\tau+n,\,\varepsilon-i\tau\right)=\frac{\Gamma(\varepsilon+i\tau+n\,)\Gamma(\varepsilon-i\tau)}{\Gamma(2\varepsilon+n)}. \tag{3}$$

Using the next property of the Gamma function: $\Gamma(z+1)=z\Gamma(z)$, expression (3) can be rewritten as follows:

$$B\left(\varepsilon+i\tau+n,\,\varepsilon-i\tau\right)=f(\varepsilon,\tau)\left(\frac{\varepsilon}{\varepsilon^2+\tau^2}+i\frac{\tau}{\varepsilon^2+\tau^2}\right), \tag{4}$$

where

$$f(\varepsilon,\tau)=\frac{\Gamma(\varepsilon+i\tau+n\,)\Gamma(\varepsilon-i\tau+1)}{\Gamma(2\varepsilon+n)}. \tag{5}$$

Let us show that the expression (4) and, accordingly, the sequence of analytic functions $B\left(\varepsilon+i\tau+n,\,\varepsilon-i\tau\right)$ has a weak limit. For this purpose let us study the first summand on the right hand side of formula (4).

The function $f(\varepsilon,\tau)$ is analytic at the point of the complex plane: $\varepsilon=0,\tau=0$. Using the Taylor formula with the remainder of the Lagrange form, one can expand the function mentioned above in the neighborhood of the point $(0,\ \tau)\,,\quad \tau\in\mathbb{R}$ (see, e.g., [3]):

$$f(\varepsilon,\ \tau)=f(0,\ \tau)+\frac{1}{1!}f'_{\varepsilon}(0,\ \tau)\varepsilon+\frac{1}{2!}f''_{\varepsilon\varepsilon}(\xi,\ \tau)\varepsilon^2,\quad 0<\xi<\varepsilon. \tag{6}$$

The coefficients of the decomposition have the following form:

$$\begin{aligned}
&f\left(0,\,\tau\right)=\Gamma^{-1}\left(n\right)\Gamma\left(i\tau+n\right)\Gamma\left(-i\tau+1\right),\\
&f'_{\varepsilon}\left(0,\,\tau\right)=f\left(0,\,\tau\right)\left\{\psi\left(i\tau+n\right)+\psi\left(-i\tau+1\right)-2\psi\left(n\right)\right\},\\
&f''_{\varepsilon\varepsilon}\left(\xi,\,\tau\right)=f\left(\xi,\,\tau\right)\left\{\left[\psi\left(i\tau+n\right)+\psi\left(-i\tau+1\right)-2\psi\left(n\right)\right]^2\right.\\
&\left.+\psi'\left(i\tau+n\right)+\psi'\left(-i\tau+1\right)-4\psi'\left(n\right)\right\},
\end{aligned} \tag{7}$$

where $\psi\left(x\right)$ is the digamma function

$$\psi\left(x\right)=\frac{d}{dx}\Gamma\left(x\right)=\Gamma'\left(x\right)/\Gamma\left(x\right).$$

Note, that while calculating the weak limits the coefficients (7) don't change the condition of continuity in the integrands.

Using the next representation of the Dirac function (see, e. g., [4], p. 54)

$$\delta(\tau)=\lim_{\varepsilon\to 0+}\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+\tau^2},$$

which has the following integral form:

$$\int\limits_{-\infty}^{+\infty}d\tau\varphi(\tau)\delta(\tau)=\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty}d\tau\varphi(\tau)\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+\tau^2}=\varphi\left(0\right),$$

and the formulas (6) and (7), one can show that:

$$\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty} d\tau\varphi\left(\tau\right)f(\varepsilon,\tau)\frac{\varepsilon}{\varepsilon^2+\tau^2}=\pi\varphi\left(0\right) \tag{8}$$

for any continuous and bounded $\varphi\left(\tau\right)$.

Similarly, for the second summand of expression (4), we obtain:

$$\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty} d\tau\varphi\left(\tau\right)f(\varepsilon,\tau)\frac{i\tau}{\varepsilon^2+\tau^2}=\frac{i}{2}\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty} d\tau\varphi\left(\tau\right)f(0,\tau)\left\{\frac{1}{\tau+i\varepsilon}+\frac{1}{\tau-i\varepsilon}\right\}. \tag{9}$$

Substituting the Sokhotsky formula (see, e. g., [4], p. 52) in (9), one can show that:

$$\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty} d\tau\varphi\left(\tau\right)f(\varepsilon,\tau)\frac{i\tau}{\varepsilon^2+\tau^2}=\int\limits_{-\infty}^{+\infty} d\tau\frac{g\left(\tau\right)}{\tau}, \tag{10}$$

where according to the first formula of (7)

$$g\left(\tau\right)=i\varphi\left(\tau\right)f(0,\tau)=i\Gamma^{-1}\left(n\right)\varphi\left(\tau\right)\Gamma\left(i\tau+n\right)\Gamma\left(-i\tau+1\right).$$

The integral on the right-hand side of (10) is regarded in the sense of the principal value Cauchy.

Due to equalities (8) and (10), one obtains the weak limit we are looking for:

$$\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{+\infty} d\tau\varphi\left(\tau\right)B\left(\varepsilon+i\tau+n,\ \varepsilon-i\tau\right)=\pi\varphi\left(0\right)+\int\limits_{-\infty}^{+\infty} d\tau\frac{g\left(\tau\right)}{\tau}. \tag{11}$$

Expression (11) has the following formal form:

$$\lim_{\varepsilon\to 0+}B\left(\varepsilon+i\tau+n,\,\varepsilon-i\tau\right)=\pi\delta(\tau)+\frac{i}{\Gamma\left(n\right)}\Gamma\left(i\tau+n\right)\Gamma\left(-i\tau+1\right)\frac{P}{\tau}, \tag{12}$$

where P/τ denotes the Cauchy principle value.

Using the conjugate of expression (12) and the next property $B\left(\alpha,\,\beta\right)=B\left(\beta,\,\alpha\right)$ of the beta function, one can derive:

$$\lim_{\varepsilon\to 0+}B\left(\varepsilon+i\tau,\,\varepsilon-i\tau+n\right)=\pi\delta(\tau)-\frac{i}{\Gamma\left(n\right)}\Gamma\left(n-i\tau\right)\Gamma\left(i\tau+1\right)\frac{P}{\tau}. \tag{13}$$

According to the formula (see, e. g., [5], p. 82)

$$\left|\Gamma\left(1+i\tau\right)\right|^2=\frac{\pi\tau}{\sinh\left(\pi\tau\right)},$$

for $n=1$ from equalities (12) and (13) one obtains:

$$\begin{aligned} &\lim_{\varepsilon\to 0+} B\left(\varepsilon+i\tau+1,\,\varepsilon-i\tau\right)=\pi\delta(\tau)+i\pi\sinh^{-1}\left(\pi\tau\right),\\ &\lim_{\varepsilon\to 0+} B\left(\varepsilon+i\tau,\,\varepsilon-i\tau+1\right)=\pi\delta(\tau)-i\pi\sinh^{-1}\left(\pi\tau\right). \end{aligned} \tag{14}$$

By virtue of formulas (14), in the neighborhood of the point $\varepsilon=0,\ \tau=0$ of the complex plane, one can write:

$$\begin{aligned} \lim_{\varepsilon\to 0+} B\left(\varepsilon+i\tau+1,\,\varepsilon-i\tau\right)&=\frac{i}{\tau+i0},\\ \lim_{\varepsilon\to 0+} B\left(\varepsilon+i\tau,\,\varepsilon-i\tau+1\right)&=-\frac{i}{\tau-i0}, \end{aligned}$$

where the functions on the right-hand side are proportional to the Sokhotsky functions.

3 Conclusions. We showed that the limiting behaviour $\varepsilon\to 0+$ of the weak sequences of the analytic functions $B\left(\varepsilon+i\tau+n,\,\varepsilon-i\tau\right)$ one can express by the singular generalized functions, in particular, by the Dirac and Sokhotsky functions.

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