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SUMMABILITY OF FOURIER SERIES FOR ALMOST PERIODIC ON LOCALLY COMPACT ABELIAN GROUPS FUNCTIONS WITH VALUES IN BANACH SPACES

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Abstract. The problem of summability of Fourier series of continuous almost periodic on locally compact Abelian groups functions with values in Banach spaces is considered.

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Consider a locally compact Abelian group G with a Haar measure μ, assuming that the topology of G is Hausdorff. We denote by $\widehat{G}$ the group dual to G, i.e., the group of all continuous homomorphisms (characters) acting from G to the group which is the unit circle endowed with the topology of uniform convergence on compact subsets of G.

In the well-known paper of S.Bochner and J.von-Neumann [1], the notion of almost periodic functions, defined on arbitrary groups with values in topological spaces X is introduced. In the case of a locally compact Abelian group G and Banach space X, this definition takes the following form

Definition 1. Let f be a continuous function defined on a locally compact Abelian group G and having the values in a Banach space X. The function f is called almost periodic on G, if for a fixed $x \in G$ and arbitrary $g \in G$, from the family $\{f(xg)\}$ one can choose an uniformly convergent with respect to x subsequence in the sense of the norm.

One can verify that if G is the group of real numbers, X is the Banach space of real numbers and f satisfies the condition of Definition 1, then it is H.Bohr's classical almost periodic function [2].

The set of all almost periodic on G functions with values in a Banach space X will be denoted by $AP(G,X)$. In [1] it is proved that $AP(G,X)$ is the linear closed set in X. Moreover, every $f \in AP(G,X)$ is bounded and uniformly continuous on G. If C is the Abelian group of the complex numbers, $f \in A_P(G,X)$ and $\alpha \in AP(G,C)$, then $\alpha f \in AP(G,X)$.

Important is the proven in [1] following theorem on the mean value of almost periodic functions.

Theorem 1. *Let G be a topological group, let X be a Banach space and a function f belongs to the space $AP(G,X)$. Then there exists a sequence of systems* $(n = 1, 2, \cdots)$ $a_{n,1}, \cdots,$ $a_{n,m_n} \in G$, $\alpha_{n,1}, \cdots, \alpha_{n,m_n} \geq 0, \alpha_{n,1} + \cdots + \alpha_{n,m_n} = 1$ *and an unique element $\psi \in X$ such that the sequence*

$$\{\alpha_{n,1} f(a_{n,1}g) + \cdots + \alpha_{n,m_n} f(a_{n,m_n}g),\ g \in G\}$$

converges to a ψ in the sence of the norm of X, when $n \to \infty$, uniformly with respect to g.

For the theory of almost periodic functions it is important that for arbitrary $f \in AP(G,X)$ the mean value $M_g\{f(g)\overline{\chi(g)}\}$ of the function $f(g)\overline{\chi(g)}$, $\chi \in \widehat{G}$, is nonzero at most on countable sets of characters χ (the subscript g at M means that the mean value is taken relative to g). This circumstance makes it possible that each function $f \in AP(G,X)$ is brought into correspondence with the formal Fourier series

$$f(g) \sim \sum_{n=0}^{\infty} a_n(f)\chi_n(g),\ \chi_n \in \widehat{G},\ \chi_0 = e, \tag{1}$$

where e is the unit element of the group $\widehat{G}$ and the coefficients $a_n(f) = M_g\{f(g)\overline{\chi(g)}\}$ are elements of the space X.

We study the problem of summability of series (1) in the case when $\{\chi_n\}$ has unique accumulation point at the infinity of $\widehat{G}$. In such a case, we denote by $P_K(f,X)$ the set of elements of the form $\sum_{\chi_k \in K} a_k\chi_k(g)$, where K is a compact set of $\widehat{G}$, a_k are elements of X and $g \in G$. The elements of $P_K(f,X)$ are called trigonometric polynomials of degree K.

Let $L(G)$ be the space of integrable on G by a Haar measure μ functions. For a function $f \in L(G)$ we can define its Fourier transform $\widehat{f}$ on $\widehat{G}$ according to the equality $\widehat{F}(\chi) = \int_G f(g)\overline{\chi(g)}d_\mu g,\ \chi \in \widehat{G}$. We denote by $U_{\widehat{G}}$ the collection of all symmetric compact sets from $\widehat{G}$ which are closures of neighborhoods of the unit element of $\widehat{G}$. For a $T \in U_{\widehat{G}}$ we consider a defined on $\widehat{G}$ real valued continuous function φ_T such that $\mathrm{supp}\varphi_T \subset T$, the Fourier transform $V_T(g) := \widehat{\varphi_T}(g)$ belongs to $L(G)$ and $\int_G V_T(g)dg = 1$. For $f \in A_P(G,X)$ and V_T we define the following function

$$f_T(g) := \int_G f(gs)V_T(s)ds,\ g \in G. \tag{2}$$

The integral in (2) is understood in the Bochner sense [3].

Proposition 1. *Let f be a function from $AP(G,X)$, whose Fourier series has the form (1) and every $T \in U_{\widehat{G}}$ contains only a finite number of the entering in (1) characters χ_n. If the acting from G to X function f_T is defined by (2), in which the numerical function V_T satisfies the above mentioned conditions, then*

$$f_T(g) = \sum_{\chi_k \in T} a_k(f)\varphi_T(\chi_k)\chi_k(g). \tag{3}$$

Proof. We have for a fixed $g \in G$ and arbitrary $h \in G$

$$||f_T(gh)|| = ||\int_G f(sgh)V_T(s)ds|| \le \int_G ||f(sgh)|| \cdot |V_T(s)|ds. \tag{4}$$

Since f belongs to the space $AP(G, X)$, from the family $\{f(sgh)\}$ one can select a subsequence $\{f(sgh_k)\}$, $h_k \in G$, which converges in X uniformly with respect to $s^{-1}g$. Therefore, from (4) it follows that the function f_T belongs to $AP(G, X)$. Let us calculate the Fourier coefficient b_k of the function f_T which corresponds to the character χ_k. Namely

$$b_k(f_T) = M_g\{f_T(g)\overline{\chi_k(g)}\} = M_g\{\int_G V_T(s)\chi_k(s)f(sg)\overline{\chi_k(gs)}ds\}.$$

By application of Theorem 1 we see that the coefficient $b_k(f_T)$ can be represented in the following form

$$b_k(f_T) = \lim_{n\to\infty}\sum_{i=1}^{m_n}\int_G \alpha_{n,i}V_T(s)\chi_k(s)\overline{f(a_{n,i}gs)}\chi_k(a_{n,i}gs)ds,$$

where $\sum_{i=1}^{m_n}\alpha_{n,i} = 1$ and $a_{n,i}$ $i = 1, \cdots, m_n$ are some elements of the group G. One can pass to the limit inside the integral, because $V_T \in L(G)$ and f is bounded on X. Applying well-known properties of mean values ([1], p. 29), we get

$$b_k(f_T) = \int_G V_T(s)\chi_k(s)M_g\{f(gs)\overline{\chi_k(gs)}\}ds$$

$$= \int_G V_T(s)\chi_k(s)M_g\{f(g)\overline{\chi_k(g)}\}ds = \widehat{V_T}(\chi_k^{-1})M_g\{f(g)\overline{\chi_k(g)}\} = a_n(f)\varphi_T(\chi_k). \quad (5)$$

If $\chi_k \overline{\in} T$, we have from (5) that $b_k = 0$. From this and (5) it follows that the right hand of (3) is the Fourier series of the function $f_T(g)$. From the uniqueness theorem of Fourier series for almost periodic functions ([1], p.36) follows equality (3). □

For the case when $G = R$, $X = C$, Proposition 1 was proved in [4].

Theorem 2. *Let G be a locally compact Abelian group, $T \in U_{\widehat{G}}$, $V_T = \widehat{\varphi_T}$, $\varphi(e) = 1$, $\lim_{T\to\widehat{G}}\varphi_T(g) = 1$ for arbitrary fixed $g \in G$ and the integrals $\int_G |V_T(g)|dg$ are uniformly bounded according to T. Let us assume moreover that for $f \in AP(G, X)$ the conditions from Proposition 1 are satisfied. Then uniformly with respect to $g \in G$*

$$\lim_{T\to\widehat{G}} f_T(g) = f(g),$$

where $f_T(g)$ is the trigonometric polynomial of the degree T, defined by (3).

Proof. According to the well-known approximate theorem ([1], p.37), for the function $f \in AP(G, X)$ and $\varepsilon > 0$ there exists a trigonometric polynomial $Q_{T_0}(g) = \sum_{\chi_k\in T_0} c_k\chi_k(g)$, $c_k \in X$ such that

$$||f - Q_{T_0}|| < \varepsilon. \quad (6)$$

We can see this also by a simple generalization of the proof, given in [4] or in [5]. It is important to note that the characters appearing in a finite polynomial Q_{T_0} are chosen from the characters encountered in the Fourier series of the function $f(g)$. Based on (6), we have for arbitrary symmetric with respect to the unity e compact set $T \supset T_0$ of $\widehat{G}$ that

$$|| \int_G Q_{T_0}(gu)V_T(u)du - \int_G f(gu)V_T(u)du|| < \varepsilon \int_G |V_T(u)|du. \tag{7}$$

But $\int_G Q_{T_0}(gu)V_T(u)du = \sum_{\chi_k \in T_0} c_k \int_G \chi_k(gu)V_T(u)du = \sum_{\chi_k \in T_0} c_k \varphi_T(\chi_k)\chi_k(g)$ and we have from (7)

$$|| \sum_{\chi_k \in T_0} c_k \varphi_T(\chi_k)\chi_k(g) - f_T(g)|| < \varepsilon \int_G |V_T(u)|du. \tag{8}$$

From the conditions imposed on the function φ_T it follows that we can choose such compact $T \subset \widehat{G}$ for which the following holds

$$|1 - \varphi_T(\chi_k)| < (\sum_{\chi_k \in T_0} ||c_k||)^{-1}\varepsilon.$$

Then we have

$$||Q_{T_0}(g) - \sum_{\chi_k \in T_0} c_k \varphi_T(\chi_k)\chi_k(g)|| = || \sum_{\chi_k \in T_0} c_k(1 - \varphi_T(\chi_k)\chi_k(g)|| < \varepsilon. \tag{9}$$

It follows from (7), (8), (9) that for arbitrary set $T \in U_{\widehat{G}}$ the following is valid

$$||f - f_T|| < \varepsilon(2 + \int_G |V_T(u)|du).$$

If $T \to \widehat{G}$, than we have the validity of Theorem 2. □

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