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ON THE STRONG UNIQUENESS OF ELEMENTARY VOLUMES ON $\mathbf{R}^2$

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Abstract. The paper is concerned with some properties of the theory of elementary volume. It is shown that there exists an extension of the standard Jordan measure of $\mathbf{R}^2$, which does not possess the strong uniqueness property in the class of π_2-volumes.

Keywords and phrases: Elementary volume, the uniqueness property of volume

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1 Introduction. The deep notion of volume type functionals is closely tied with several interesting and important geometric topics, such as the uniqueness property of volumes. It is well known that the uniqueness property for volumes plays a significant role in various questions of Euclidean geometry. For instance, in general, in theory of invariant measures the standard Lebesgue measure in Euclidean spaces has the uniqueness property and this fact implies many important consequences in various directions of contemporary mathematics. In this connection see, for example [2], [4], [7].

The main purpose of this paper is to consider the uniqueness property of volumes from the general point of view and to investigate the behaviour of this property under various transformation groups of Euclidean spaces. In particular, the present paper is devoted to some aspects highlighting profound connections between elementary theory of volume with general methods of the theory of invariant measures.

2 Content. Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathrm{dom}(\mu)$ is the domain of a given measure μ;

$\mathbf{R^n}$ is the n-dimensional Euclidean space.

D_n is the group of all isometric transformations of $\mathbf{R}^n$;

π_n is the group of all translations in $\mathbf{R}^n$.

Let S_n be the semi-ring generated by the collection of all coordinate parallelepipeds of $\mathbf{R}^n$.

Let $G \subset D_n$. A functional V_n is called an elementary G-volume on $\mathbf{R}^n$ if the following four conditions hold:

(1) V_n is non-negative:

$$(\forall X)(X \in S_n \Rightarrow V_n(X) \geq 0);$$

(2) V_n is additive:

$$(\forall X)(\forall Y)(X \in S_n \wedge Y \in S_n \wedge X \cap Y = \emptyset \Rightarrow V_n(X \cup Y) = V_n(X) + V_n(Y);$$

(3) V_n is G-invariant:

$$(\forall g)(\forall X)(g \in G \wedge X \in S_n \Rightarrow V_n(g(X)) = V_n(X));$$

(4) $V_n(\triangle_n) = 1$, where $\triangle_n = [0,1]^n$ denotes the unit coordinate cube in $\mathbf{R}^n$.

The above-mentioned conditions are usually treated as Axioms of Invariant Finitely Additive Measure.

If condition (2) is replaced by the countable additivity condition, then we obtain the definition of a G-measure.

Notice that, the class of all sets, which may be covered by a finite union of sets in S_n is a ring. The ring $\mathbf{S_n}$ generated by S_n is called the class of elementary figures. A element $X \in \mathbf{S_n}$ is called the elementary figure.

It is well known that the classical Jordan measure on $\mathbf{R}^n$ is a natural example of G-volume in $\mathbf{R}^n$. Respectively, a certain extension of Jordan measure to a sufficiently large class of subsets of $\mathbf{R}^n$ is the standard Lebesgue measure. In some sense, the latter class of sets is maximal, because within the framework of constructive methods it is impossible to further enlarge this class.

From the measure theory point of view there are many interesting and important facts concerning G-volumes. The most famous among them is due to Banach.

Theorem 1. *In the case $n = 1$ and $n = 2$ there exists a non-negative additive functional defined on the family of all bounded subsets of the Euclidean space $\mathbf{R}^n$, invariant under the group of all isometries of $\mathbf{R}^n$ and extending the Lebesgue measure λ_n.*

The proof of Theorem 1 can be found in [4], [10].

Let $\mathbf{R}^n$ be again the Euclidean space, let G be a group of all isometric transformations of $\mathbf{R}^n$ and let M be a class of all G-volumes on $\mathbf{R}^n$.

We say that a volume $V_1 \in M$ has the uniqueness property if, for every volume $V_2 \in M$ such that $dom(V_1) = dom(V_2)$ we have

$$V_1 = V_2.$$

If the group G is sufficiently rich, then all elementary volumes on $\mathbf{R}^n$ coincide with each other. In particular, if G contains an everywhere dense set of translations of $\mathbf{R}^n$, then all elementary volumes coincide with the restriction of the Jordan measure to S_n.

We say that a figure $X \subset \mathbf{R}^n$ has the uniqueness property in M, if for every two volumes $V_1 \in M$ and $V_2 \in M$ such that $X \in dom(V_1) \cap dom(V_2)$ we have the equality

$$V_1(X) = V_2(X).$$

We say that a volume $V \in M$ has the strong uniqueness property in M if each figure $X \in dom(V)$ has the uniqueness property on M.

In other words, a volume $V \in M$ possesses the strong uniqueness property with respect to M if, for every $X \in dom(V)$ and for every two volumes $V_1 \in M$ and $V_2 \in M$ such that $X \in dom(V_1)$ and $X \in dom(V_2)$ we have the following equality

$$V_1(X) = V_2(X).$$

The above-mentioned definitions can be found in [2], [3], [4], [7].

It is clear that if a G-volume has the strong uniqueness property then the same volume has the uniqueness property too.

From the definition of G-volumes it follows that the unit cube on $\mathbf{R}^n$ has the uniqueness property on M. It is well known that every subset $X \subset \mathbf{R}^n$, which is measurable with respect to the classical Jordan measure, has the strong uniqueness property in the class of π_n-volumes.

It is natural to investigate the above-mentioned question of bounded subsets in $\mathbf{R}^n$, assuming that all them are measurable with respect to the Lebesgue sense.

We need several auxiliary propositions.

Lemma. *Let $\mathbf{R}^n$ be the n-dimensional Euclidean space, let j_n be the Jordan measure on $\mathbf{R}^n$ and let $\{X_i : i \in I\}$ be an arbitrary family of bounded and nowhere dense subsets of $\mathbf{R}^n$. Then there exists a measure j'_n on $\mathbf{R}^n$ extending j_n and satisfying the following relations:*

$$\{X_i : i \in I\} \subset dom(j'_n);$$

$$(\forall i)(i \in I \Rightarrow j'_n(X_i) = 0).$$

The following proposition is valid.

3 Compilation. It is well known that a Jordan curve is a homeomorphic image of the unit circle. It is well known that there exists a Jordan curve L in the Euclidean plane $\mathbf{R^2}$ possessing a positive two-dimensional Lebesgue measure, i.e., $\lambda_2(L) > 0$. The construction of L can be done directly and can be found in [5], [6], [9].

Another idea is to derive from the Denjoy-Riesz theorem that each compact zero-dimensional set C in $\mathbf{R^2}$ is contained in a Jordan curve. Taking as C a Cantor-type set in $\mathbf{R^2}$ with $\lambda_2(C) > 0$, we get the desired curve L (see, [1]).

Theorem 2. *The Jordan curve L does not have the uniqueness property with respect to the class of all D_2-volumes of $\mathbf{R^2}$.*

From Theorem 2 the next statement follows.

Theorem 3. *There exists an extension of the Jordan measure j_2 of $\mathbf{R}^2$, which does not possess the strong uniqueness property in the class of π_2-volumes.*

4 Conclusions. It is well known that the classical Jordan measure in $\mathbf{R^n}$ has the strong uniqueness property. But there exists an extension of the Jordan measure, which has not the same property with respect to the certain class of volumes.

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