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THE R-COMMUTANT AND ABELIAN VARIETIES OF EXPONENTIAL MR-GROUPS

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Abstract. In the present paper some problems of the theory of the varieties of exponential MR-groups are considered.

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The notion of exponential R-group (R is an arbitarry associative ring with identity 1) was introduced by Lyndon in [1]. Myasnikov and Remeslennikov introduced in [2] a new category of exponential R-groups (MR-groups) as a natural generalization of the notion of R-module to a noncommutative case. Recall the basic definitions (see [1, 2]).

Let $L = \langle \cdot, ^{-1}, e\rangle$ be the group language (signature); here, $\cdot$ denotes the binary operation of multiplication, $^{-1}$ denotes the unary operation of inversion, and e is a constant symbol for the identity element of the group.

We enrich the group language to the language $\mathfrak{L}^*_{gr} = \mathfrak{L}_{gr} \cup \{f_\alpha(g) \mid \ \alpha \in R\}$, where $f_\alpha(g)$ is a unary algebraic operation.

Definition 1 ([1]). A Lyndon R-group is a set G an which operations, $\cdot$, $^{-1}$, e and $\{f_\alpha(g) \mid \ \alpha \in R\}$ are defined and the following axioms hold:

(i) the group axioms;

(ii) for all $g, h \in G$ and all elements $\alpha, \beta \in R$,

$$g^1 = g, \ \ g^0 = e, \ \ e^\alpha = e; \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha \cdot g^\beta, \ \ g^{\alpha\beta} = (g^\alpha)^\beta; \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

For brevity, in the formulas expressing the axioms, we write $f_\alpha(g)$ instead of g^α for $g \in G$ and $\alpha \in R$.

Let $\mathfrak{L}_R$ denote the category of all Lyndon R-groups. Since the axioms given above are universal axioms of the language $\mathfrak{L}^*_{gr}$, it follows that $\mathfrak{L}_R$ is a variety of algebraic systems in the language $\mathfrak{L}^*_{gr}$; therefore, general theorems of universal algebra allow us to consider the varieties of R-groups, R-homomorphisms, R-isomorphisms, free R-groups, and so on.

MR**-exponential groups.** There exit Abelian Lyndon R-groups which are not R-modules (see [3], where the structure of a free Abelian R-group was studied in detail). The authors of [1] augmented Lyndon's axioms (quasi-identity):

$$(MR) \qquad \forall_{g,h\in G},\ \ \alpha \in R\ \ [g,h]=e \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha\ \ \big([g,h]=h^{-1}h^{-1}gh\big). \qquad (4)$$

Definition 2 ([2]). An MR-group is a group G on which the operations g^α are defined for all $g \in G$ and $\alpha \in R$ so that axiom (1)–(4) hold.

Let $\mathfrak{M}_R$ denote the class of all R-exponential groups with axioms (1)–(4). Clearly, this class is a quasi-variety in the language $\mathfrak{L}^*_{gr}$, and free MR-groups, MR-homomorphisms, and so on are defined; moreover, each Abelian MR-group is an R-module and vice versa.

Most of natural examples exponential groups belong to the class $\mathfrak{M}_R$. For example, unipotent groups over a field K of zero characteristic are MK-groups, pro-p-groups are exponential groups over the ring of p-adic integers, etc (see [2] for examples).

A systematic study of MR-group was initiated in [4–10]. Results obtained in these papers have turned out to be very useful in solving well-known problems of Tarski.

Below, following [2], we recall some definitions in the category of MR-groups. Let G be an MR-group.

Definition 3 ([2]). A homomorphism of R-groups $\varphi : G_1 \to G_2$ is called an R**-homomorphism** if $\varphi(g^\alpha) = \varphi(g)^\alpha$, $g \in G$, $\alpha \in R$.

Definition 4 ([2]). For $g, h \in G$ and $\alpha \in R$, the element

$$(g,h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$$

is called the α**-commutator** of the elements g and h.

It is obvious that for $\alpha = -1$ the α-commutator $(g,h)_\alpha$ coincides with the usual commutator $[h^{-1}, g^{-1}]$.

Clearly, $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in \mathfrak{M}_R \iff \big([g,h]=e \Longrightarrow (g,h)_\alpha = e\big)$. This equivalence leads to the definition of an $\mathfrak{M}_R$-ideal.

Definition 5 ([2]). A normal R-subgroup $H \trianglelefteq G$ is called an $\mathfrak{M}_R$-ideal if, $(g,h)_\alpha \in H$ for all $g \in G$, $h \in H$ and $\alpha \in R$.

Proposition 1 ([2]).

(i) *If $\varphi : G_1 \to G_2$ is an R-homomorphism in the category $\mathfrak{M}_R$-groups, then* $\ker\varphi$ *is an $\mathfrak{M}_R$-ideal in G.*

(ii) *If H is an $\mathfrak{M}_R$-ideal in G, then $G/H \in \mathfrak{M}_R$.*

Varieties of exponential MR-group. Varieties are closely related to free groups, since identities are the elements of free groups. Let $X = \{x_i \mid \ i \in I\}$ be an infinite aplhabet and $F_R(X)$ be a free MR-group with free generating set X as an MR-group. Let us call arbitrary element $w(x_1, \dots, x_n) \in F_R(X)$ **R-word** in X. Let G be an MR-group and $g_1, \dots, g_n \in G$. The map $x_i \mapsto g_i$ can be extended to an R-homomorphism $\varphi : F_R(X) \to G$. The image of the word $w(x_1, \dots, x_n)^\varphi \in G$ under this homomorphism is called value of $w(x_1, \dots, x_n)$ on the elements $g_1, \dots, g_n$. Fix the following notations:

$$w(x_1, \dots, x_n) = w(\overline{x}), \ \ \overline{x} = (x_1, \dots, x_n), \quad w(g_1, \dots, g_n) = w(\overline{g}), \ \ \overline{g} = (g_1, \dots, g_n),$$
$$w(G) = \big\{w(\overline{g}) \mid \ \overline{g} \in G^n\big\} = \big\{w(g_1, \dots, g_n) \mid \ g_i \in G\big\}.$$

Definition 6. An R-word $w(\overline{x})$ is called an **identity on MR-group** G if $w(G) = e$.

Definition 7. Let W be a subset of $F_R(X)$. Then W defines the **variety of MR-groups**

$$\mathfrak{N} = \mathfrak{N}(W) = \big\{G \in \mathfrak{M}_R \mid \ \ w(G) = e \ \ \forall\, w \in W\big\}.$$

Definition 8. An R-word $u(\overline{x}) \in F_R(X)$ is called a **corollary** of the set of words W, if $u(G) = e$ for any group $G \in \mathfrak{N}$.

Definition 9. The $\mathfrak{M}_R$-ideal of G generated by all values of all words from W is called W-verbal ideal of G. Let us denote by $W(G)$ the W-verbal ideal of G.

Proposition 2. *A verbal ideal in $F_R(X)$ generated by the set of word W consists exactly of all corollaries of the set W in $F_R(X)$.*

Definition 10. A group $F_{W,R}(X) \in \mathfrak{N}$ is called a *free group with the base X in the varieties* $\mathfrak{N}$ if $F_{W,R}(X)$ R-generated by the set X and for any group $G \in \mathfrak{N}$ arbitrary map $\varphi_0 : X \to G$ can be extended to an R-homomorphism $\varphi : F_{W,R}(X) \to G$.

Theorem 1. *The group $F_{W,R}(X)/W(F_{W,R}(X))$ is a free group in the varieties of $\mathfrak{N}$ which is defined by W.*

Definition 11. The subgroup $(G, G)_R = \langle (g, h)_\alpha \mid \ g, h \in G, \ \alpha \in R\rangle_R$ of G is called the R-**commutant** of G.

Theorem 2. *For any MR-group G the following is true:*

(i) *The R-commutant of G is the verbal MR-subgroup defined by the word $[x, y] = x^{-1}y^{-1}xy$.*

(ii) *The R-commutant is the smallest $\mathfrak{M}_R$-ideal whose factor-group is abelian.*

Let us describe abelian varieties of exponential MR-groups. To this end, let us firstly determine the structure of a free abelian exponential MR-group.

Theorem 3. *Each free abelian MR-group with base X is a free R-module and R-isomorphic to a factor-group of a free MR-group with base X by its R-commutant.*

Theorem 4. *There is a one-to-one correspondence between the lattice of two-sided ideals of a ring R and the lattice of verbal MR-subgroups of a free R-module.*

Corollary. *If $R = \mathbb{Z}$ then each proper subvariety of abelian groups of period n, $n \geq 2$.*

Remark. When defining the varieties of $\mathfrak{M}_R$-groups we follow V. A. Gorbunov's monograph [11], which declares how one can understand varieties of groups inside, quasi-varieties of groups. Therein it is shown that for these varieties all the well-known Birkhoff theorems are that for them there exist the notion of a free group and the theory of defining relations.

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