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OBTAINING NUMERICAL-ANALITICAL SOLUTIONS OF THE CONTACT PROBLEMS ELAATICITY FOR SOME DISTRIBUTED LOADS

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Abstract. The paper consider the normal contact problems, which are formulated as follows: the block, weight which can be neglected, to press with a certain force on the surface of a half-space, i.e. at the contact surface is given normal stress (is given contractive stress). We examine two types of distributed load, which correspond to the following cases: a) when contact surface is flat and b) when the contact surface is of parabolic shape. The work considers plane deformation state. The stress-strain state of an isotropic homogeneous elastic half plane is studied. The problems are solved by the boundary element method, which is based on the solutions of the problems of Flamant and Boussinesq.

Keywords and phrases: Contact problem, boundary element method, Flamant problem, Boussinesq's problem.

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1 Introduction. Many scientific works, for example [1–3], and others, devoted to contact problem have investigated the problem of the indentation of an elastic body (for instance, a stamp problem) into an elastic foundation in the form of a halfspace with a constant modulus of elasticity (exact solutions), or they have investigated the indentation into an elastic homogenous layer (approximate solutions). In the author's earlier works [4-6] the boundary value problems and the boundary contact problems are solved by BEM based on the singular solution of the Kelvin problem.

In the present paper we will consider the normal contact problems, which is so formulated: indentor, whose weight may be neglected, pressed with certain force on the surface half space, that is normal stress acts at the contact surface (here considered are compressible stresses and in the case of the stretching tension will be opening the contact or cracks which here we do not consider), and the tangential stress is zero. In particular, we consider two types of distributed load, which correspond to the following cases: a) when the half-space subjected to frictionless flat rigid indenter, b) when the half-space subjected to frictionless cylindrical rigid indenter. The article consider the plane deformation. Problems are solved by the boundary element method, which is based on solutions of problems Flamant (BEMF) and Boussinesq (BEMB).

In this article the stress-strain state of an isotropic homogeneous elastic half plane is studied. The obtained results by BEMF and BEMB are compared.

2 Contact problems. Assume that the indentor, whose can be neglected, relies on the half-space. Implied that the indentor and half-space have identical elastic characteristics are in state of plane deformation. On the indenter vertical load acts. We ought to find distribution of the stresses on the half plane. It is the problem press of rigid indenter on the half-plane with lubrication at contact. Lubrication provides the conditions under which tangential stress does not occur at contact of the indenter with the half-plane. Thus, we ought to solve the equilibrium equations of an isotropic homogeneous elastic body free of volume forces [7] with some boundary conditions.

Consider the problem for the half plane, when at the entire border of the tangent stress, and at the part of boundary $|x| > c,\ y = 0$ the normal stress are equal to zero. On the segment $|x| \le c,\ y = 0$ known normal stress σ_{yy}. So, we will find the solution of equilibrium of the equations system [7], that satisfy the following boundary conditions:

$$\begin{cases} \sigma_{yx} = 0, & |x| < \infty, \quad y = 0, \\ \sigma_{yy} = 0, & |x| > c, \quad y = 0, \\ \sigma_{yy} = -p(x), & |x| \le c, \quad y = 0. \end{cases} \tag{1}$$

We will solve the contact problems (1) for the indenter with flat and parabolic contour by the BEMF and BEMB.

By using solution of the **Flamant problem**, for the problem (1) numerical values of stresses and displacements in points $\left(x^i, y^k\right)$ of the half-plane receive by following equalities:

$$\begin{aligned}
\sigma_{xx}^{i(F)}\left(x^i, y^k\right) &= \sum_{j=1}^{N} A_{xx}^{ij(F)} p\left(x_j\right), \\
\sigma_{yy}^{i(F)}\left(x^i, y^k\right) &= \sum_{j=1}^{N} A_{yy}^{ij(F)} p\left(x_j\right), \\
\sigma_{xy}^{i(F)}\left(x^i, y^k\right) &= \sum_{j=1}^{N} A_{xy}^{ij(F)} p\left(x_j\right), \\
u_{x}^{i(F)}\left(x^i, y^k\right) &= \sum_{j=1}^{N} B_{x}^{ij(F)} p\left(x_j\right), \\
u_{y}^{i}\left(x^i, y^k\right) &= \sum_{j=1}^{N} B_{y}^{ij} p\left(x_j\right),
\end{aligned}$$

where N is a number of boundary element, $A_{xx}^{ij(F)}$, $A_{yy}^{ij(F)}$, $B_{y}^{ij(F)}$ are the boundary coefficients of stress influence, which are received from the solution of Flamant problem. For example, coefficient $A_{yy}^{ij(F)}$ gives the normal stress at point $\left(x^i, y^k\right)$, which is induced by the constant unit normal stress acting at the jth element, and

$$A_{yy}^{ij(F)} = \frac{1}{\pi}\left[\left(\arctan\frac{y^k}{x^i-x^j-a^j} - \arctan\frac{y^k}{x^i-x^j+a^j}\right) - \frac{y^k\left(x^i-x^j+a^j\right)}{(x^i-x^j+a^j)^2+\left(y^k\right)^2} + \frac{y^k\left(x^i-x^j-a^j\right)}{(x^i-x^j-a^j)^2+\left(y^k\right)^2}\right].$$

By using solution of the **Boussinesq problem**, for problem (1) numerical values of stresses and displacements in points $\left(x^i, y^k\right)$ of the half-plane are obtained by the following equalities:

$$\begin{aligned}
\sigma_{xx}^{i(B)}\left(x^i,y^k\right) &= \sum_{j=1}^{N} A_{xx}^{ij(B)} p\left(x_j\right),\\
\sigma_{yy}^{i(B)}\left(x^i,y^k\right) &= \sum_{j=1}^{N} A_{yy}^{ij(B)} p\left(x_j\right),\\
\sigma_{xy}^{i(B)}\left(x^i,y^k\right) &= \sum_{j=1}^{N} A_{xy}^{ij(B)} p\left(x_j\right),\\
u_{x}^{i(B)}\left(x^i,y^k\right) &= \sum_{j=1}^{N} B_{x}^{ij(B)} p\left(x_j\right),\\
u_{y}^{i(B)}\left(x^i,y^k\right) &= \sum_{j=1}^{N} B_{y}^{ij(B)} p\left(x_j\right),
\end{aligned}$$

where $A_{xx}^{ij(B)}$, $A_{yy}^{ij(B)}$, $B_{y}^{ij(B)}$ are the boundary coefficients of stress influence, which are received from the solution of Boussinesq's problem. For example, coefficient $B_{y}^{ij(B)}$ gives the normal displacement at point $\left(x^i,y^k\right)$, which is induced by the constant unit normal stress acting at the jth element, and

$$\begin{aligned}
B_{y}^{ij(B)} = \tfrac{1}{4\pi\mu}\Bigg[&\frac{y^k}{\sqrt{(x^i-x^j-a^j)^2+\left(y^k\right)^2}} - \frac{y^k}{\sqrt{(x^i-x^j+a^j)^2+\left(y^k\right)^2}}\\
&+(1-2\nu)\left(\ln\frac{\sqrt{(x^i-x^j-a^j)^2+\left(y^k\right)^2}+y^k}{\sqrt{(x^i-x^j+a^j)^2+\left(y^k\right)^2}+y^k} - \ln\frac{\sqrt{(L-x^j-a^j)^2+\left(y^k\right)^2}+y^k}{\sqrt{(L-x^j+a^j)^2+\left(y^k\right)^2}+y^k}\right)\Bigg].
\end{aligned}$$

We consider task for 2-D elastic half-space subjected frictionless a) flat rigid indenter, b) parabolic-cylindrical rigid indenter.

In the case a), i.e. when along the linear loaded part of the boundary the displacement u_y is constant, at first A. Chapling, and thereupon Sadovsky's obtained the following formula of distribution contact load [8]:

$$p(x) = \frac{P}{\pi\sqrt{c^2-x^2}},$$

and in the case b), the contact load can be written in the form [8]:

$$p(x) = \frac{2P}{\pi c^2}\sqrt{c^2-x^2}.$$

Using the MATLAB software we obtained numerical results and constructed graphs of normal and tangential stresses. Their comparison shows that visually in both cases (obtained by BEMF and BEMB) looks equally. Observations showe that obtained by BEMF stresses $\frac{\sigma_{yy}}{P}$ approximate the given $\frac{\sigma_{yy}}{P}$ stresses with more accurancy.

3 Conclusions. The main results of this work can be formulated as follows:

1. Discusses in case plane deformation of the normal contact problems for homogeneous isotropic half-space for distributed normal loads which correspond to the following cases: a) when contact surface is flat and b) when the contact surface is of parabolic shape.
2. Contact problems are solved by boundary element methods, which are based on a) singular solution of a Flamant problem (BEMF), b) singular solution of Boussinesq problem (BEMB).
3. Stress-strain state of the half plane is studied.
4. Analyzed and compared the results obtained by BEMF and BEMB.

The contact problems are common in practice, e.g. in building mechanics, mechanical engineering, mining mechanics, soil mechanics, biology, medicine, etc., the study of the deformed mode of such bodies is topical and consequently, in my opinion, setting the problems considered in the article and the method of their solution is interesting in a practical view.

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