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THE NUMBER OF REPRESENTATIONS OF NUMBERS BY POSITIVE BINARY QUADRATIC FORMS WITH EVEN DISCRIMINANT

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Abstract. The formulae for the average number of representations of numbers by a genus of positive binary quadratic forms with even discriminants are given. It gives us the opportunity to obtaim the formulae for the number of representations of positive integers by some binary forms with even discriminant belonging to multi-class genera.

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Let $r(n; f)$ denote the number of representations of a natural number n by a positive integral quadratic form $f = f(x_1, x_2, \ldots, x_s)$.

It is well known that, for the case $s > 4$, $r(n; f)$ can be represented as $r(n; f) = \rho(n; f) + \nu(n; f)$, where $\rho(n; f)$ is a "singular series" and $\nu(n; f)$ is a Fourier coefficient of a cusp form. This can be expressed in terms of the theory of modular forms by stating that $\vartheta(\tau; f) = E(\tau; f) + X(\tau)$, where

$$\vartheta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f)\, Q^n, \quad \tau \in H = \{\tau : \operatorname{Im} \tau > 0\},$$

$Q = e^{2\pi i\tau}$, $X(\tau)$ is a cusp form and

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f)\, Q^n$$

is the Eisenstein series corresponding to f.

Siegel [1] proved that if the number of variables of a quadratic form f is $s > 4$, then

$$E(\tau; f) = F(\tau; f), \tag{1}$$

where $F(\tau; f)$ denote a theta-series of a genus containing a primitive integral quadratic form f. From formula (1) follows the well-known Siegel's theorem: The sum of the singular series corresponding to the quadratic form f is equal to the average number of representations of a natural number by a genus that contains the form f. Later Ramanathan [2] proved that for any primitive integral quadratic form f with $s \geq 3$ variables (except for zero forms with variables $s = 3$ and zero forms with variables $s = 4$ whose discriminant is a perfect square), there is a function $E(\tau, z; f)$, which he called the Eisenstein-Siegel series and which is regular for any fixed τ when $\operatorname{Im} \tau > 0$ and $\operatorname{Re} z > 2 - \frac{s}{2}$, analytically extendable in a neighborhood of $z = 0$, and that

$$F(\tau; f) = E(\tau, z; f)\big|_{z=0}. \tag{2}$$

For $s > 4$ the function $E(\tau, z; f)\big|_{z=0}$ coinsides with the function $E(\tau; f)$ and the formula (2) with Siegel's formula (1).

In [3] we proved that the function $E(\tau, z; t)$ is analytically extendable in a neighborhood of $z = 0$ also in the case where f is any nonzero integral binary quadratic form (both positive-definite and indefinite) and that

$$F(\tau; f) = \frac{1}{2}\, E(\tau, z; f)\big|_{z=0}. \tag{3}$$

Ramanathan [2] believed that this result fails to hold for positive and nonzero indefinite binary quadratic forms. It follows from (3) that half of "the sum of a generalized singular series" that corresponds to a binary quadratic form is equal to the average number of representations of a natural number by the genus containing this quadratic form. In particular, if a quadratic forms belongs to a one-class genus, then for natural n

$$r(n; f) = \frac{1}{2}\, \rho(n; f). \tag{4}$$

From the result of [3] and [4] we obtain that in the case of positive binary quadratic form f with even discriminant $\rho(n; f)$ can be calculated as follows.

Theorem 1. *Let $f = ax^2 + bxy + cy^2$ be a primitive positive binary quadratic form with even discriminant $d = b^2 - 4\,ac$, $(a, d) = 1$, $\Delta = -\frac{d}{4} = r^2\omega$ (ω is a square-free number), $n = 2^\alpha m$ $(2 \nmid m)$, $\Delta = 2^\gamma \Delta_1$ $(2 \nmid \Delta_1)$, $p^l \| \Delta$, $p^\beta \| n$, $u = \prod\limits_{\substack{p \mid n \\ p \nmid 2\Delta}} p^\beta$ (p is an odd prime number), then*

$$\rho(n; f) = \frac{\pi \chi_2 \prod\limits_{p \mid \Delta, p > 2} \chi_p \sum\limits_{\nu \mid u} \left(\frac{-\Delta}{\nu}\right)}{\Delta^{\frac{1}{2}} \prod\limits_{p \mid r, p > 2} \left(1 - \left(\frac{-\omega}{p}\right)\frac{1}{p}\right) L(1, -\omega)},$$

where

$$\begin{aligned}
\chi_2 &= 2^{\frac{\alpha}{2}+2} \ \ \text{if } 2|\gamma,\ \ 0 \le \alpha \le \gamma - 3,\ \ 2|\alpha,\ \ m \equiv a \,(\mathrm{mod}\, 8);\\
&= 0 \ \ \text{if } 2|\gamma,\ \ 0 \le \alpha \le \gamma - 3,\ \ 2|\alpha,\ \ 2|\gamma,\ \ m \not\equiv a \,(\mathrm{mod}\, 8) \ \ \text{or } \ 0 \le \alpha \le \gamma - 1,\ \ 2 \nmid \alpha;\\
&= \left(1 + (-1)^{\frac{1}{2}(m-a)}\right) 2^{\frac{\alpha}{2}} \ \ \text{if } 2|\gamma,\ \ \alpha = \gamma - 2;\\
&= \left(1 + (-1)^{\frac{1}{2}(m-a)}\right) 2^{\frac{\gamma}{2}} \ \ \text{if } 2|\gamma,\ \ \alpha \ge \gamma,\ \ 2|\alpha,\ \ \Delta_1 \equiv 1 \,(\mathrm{mod}\, 4);\\
&= 2^{\frac{\gamma}{2}} \ \ \text{if } 2|\gamma,\ \ \alpha = \gamma,\ \ \Delta_1 \equiv -1 \,(\mathrm{mod}\, 4);\\
&= \left(2 - (-1)^{\frac{1}{4}(\Delta_1+1)}\right) 2^{\frac{\gamma}{2}} + \left(1 + (-1)^{\frac{1}{4}(\Delta_1+1)}\right)(\alpha - \gamma - 2) 2^{\frac{\gamma}{2}-1}\\
&\qquad \text{if } 2|\gamma,\ \ \alpha > \gamma,\ \ 2|\alpha,\ \ \Delta_1 \equiv -1 \,(\mathrm{mod}\, 4);
\end{aligned}$$

$$= \Big(1 + (-1)^{\frac{1}{4}(\Delta_1 - 1) + \frac{1}{2}(m-a)}\Big) 2^{\frac{\gamma}{2}} \quad \textit{if}\ \ 2|\gamma,\ \ \alpha \geq \gamma + 1,\ \ 2 \nmid \alpha,\ \ \Delta_1 \equiv 1 \,(\mathrm{mod}\,4);$$

$$= \Big(1 + (-1)^{\frac{1}{4}(\Delta_1 + 1)}\Big)(\alpha - \gamma - 1) 2^{\frac{\gamma}{2} - 1} \quad \textit{if}\ \ 2|\gamma,\ \ \alpha \geq \gamma + 1,\ \ 2 \nmid \alpha,\ \ \Delta_1 \equiv -1 \,(\mathrm{mod}\,4);$$

$$= 2^{\frac{\alpha}{2} + 2} \quad \textit{if}\ \ 0 \leq \alpha \leq \gamma - 3,\ \ 2 \nmid \gamma,\ \ 2|\alpha,\ \ m \equiv a \,(\mathrm{mod}\,8);$$

$$= 0 \quad \textit{if}\ \ 0 \leq \alpha \leq \gamma - 3,\ \ 2 \nmid \gamma,\ \ 2|\alpha,\ \ m \not\equiv a \,(\mathrm{mod}\,8) \ \ \textit{or}$$
$$0 \leq \alpha \leq \gamma - 2,\ \ 2 \nmid \gamma,\ \ 2 \nmid \alpha;$$

$$= \Big(1 + (-1)^{\frac{1}{4}(m-a)}\Big) 2^{\frac{1}{2}(\gamma - 1)} \quad \textit{if}\ \ 2 \nmid \gamma,\ \ \alpha \geq \gamma - 1,\ \ 2|\alpha,\ \ m \equiv a \,(\mathrm{mod}\,4);$$

$$= \Big(1 + (-1)^{\frac{1}{4}(m+a) + \frac{1}{2}(m - \Delta_1 a)}\Big) 2^{\frac{1}{2}(\gamma - 1)} \quad \textit{if}\ \ 2 \nmid \gamma,\ \ \alpha \geq \gamma - 1,\ \ 2|\alpha,\ \ m \equiv a \,(\mathrm{mod}\,4);$$

$$\chi_2 = \Big(1 + (-1)^{\frac{1}{4}(m - \Delta_1 a)}\Big) 2^{\frac{1}{2}(\gamma - 1)} \quad \textit{if}\ \ 2 \nmid \gamma,\ \ \alpha \geq \gamma,\ \ 2 \nmid \alpha,\ \ m \equiv \Delta_1 a \,(\mathrm{mod}\,4);$$

$$= \Big(1 + (-1)^{\frac{1}{4}(m + \Delta_1 a) + \frac{1}{2}(m-a)}\Big) 2^{\frac{1}{2}(\gamma - 1)} \quad \textit{if}\ \ 2 \nmid \gamma,\ \ \alpha \geq \gamma,\ \ 2 \nmid \alpha,\ \ m \equiv -\Delta_1 a \,(\mathrm{mod}\,4);$$

$$\chi_p = \left(1 + \Big(\frac{p^{-\beta} n a}{p}\Big)\right) p^{\frac{1}{2}\beta}, \quad \textit{if}\ \ \ell \geq \beta + 1,\ \ 2|\beta;$$

$$= 1 - \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\frac{1}{p}\left(1 + \left(1 + \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\right)\frac{\beta - \ell}{2}\right) p^{\frac{1}{2}\ell} \quad \textit{if}\ \ \ell \leq \beta,\ \ 2|\ell,\ \ 2|\beta;$$

$$= \left(1 - \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\frac{1}{p}\right)\left(1 + \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\right)\frac{\beta - \ell + 1}{2} p^{\frac{1}{2}\ell} \quad \textit{if}\ \ \ell \leq \beta,\ \ 2|\ell,\ \ 2 \nmid \beta;$$

$$= \left(1 + \Big(\frac{p^{-\ell}\Delta}{p}\Big)^{\beta+1}\Big(\frac{p^{-(\beta+\ell)} n a \Delta}{p}\Big)\right) p^{\frac{1}{2}(\ell - 1)} \quad \textit{if}\ \ \ell \leq \beta,\ \ 2 \nmid \ell;$$

$$= 0 \quad \textit{if}\ \ \ell \geq \beta + 1,\ \ 2 \nmid \beta.$$

The values of $L(1; -\omega)$ are given in [5] (Lemma 15).

Thus, if the genus of the quadratic form f contains one class, then according to formula (4) the problem for obtaining "exact" formulas for $r(n; f)$ is solved comlytely.

Having used Gauss's theory of genera Kaplan and Williams [6] showed the existence of positive binary quadratic forms with an even discriminant which belong to multi-class genera, but for which in case of even numbers equality (4) is true.

The following three theorems show, that the problems of obtaining formulas for the number of representations of even numbers by the binary forms belonging to multi-class genera can be easily reduced to the case of one-class genera.

Theorem 2. *Let $f = ax^2 + by^2$ be a primitive positive binary form, $\Delta = ab \equiv 3 \,(\mathrm{mod}\,4)$. Then*

$$r(2^\alpha m; f) = r(2^{\alpha - 2} m; f_1),$$

where $\alpha \geq 2$, $2 \nmid m$, $f_1 = ax^2 + axy + \frac{a+b}{4} y^2$.

Theorem 3. *Let $f = ax^2 + by^2$ be a primitive positive binary form, $\Delta = ab \equiv 3 \,(\mathrm{mod}\,4)$. Then*

$$r(2^\alpha m; f) = r(2^{\alpha - 2} m; f_1),$$

where $\alpha \geq 2,\ 2 \nmid m,\ f_1 = \frac{a+b}{4}x^2 + \frac{b-a}{2}xy + \frac{a+b}{4}y^2$.

Theorem 4. *Let* $f = ax^2 + by^2$ *be a primitive positive binary form,* $\Delta = ab \equiv 0 \pmod 4$. *Then*

$$r(2^\alpha m; f) = r(2^{\alpha-2}m; f_1),$$

where $2|\alpha,\ \alpha \geq 2,\ 2 \nmid m,\ f_1 = ax^2 + \frac{b}{4}y^2$.

Applying these theorems to the discriminants $d = -144$ and $d = -152$, we obtain the following corollaries.

Corollary 1. *Let* $f_1 = x^2 + 36\,y^2,\ f_2 = 4\,x^2 + 9\,y^2,\ n = 2^\alpha m = 2^\alpha 3^\beta u$, *where* $(m,2) = (u,6) = 1$. *Then for* $\alpha \geq 1$

$$\begin{aligned} r(n; f_1) = r(n; f_2) &= \frac{1}{2}\left(1 + \left(\frac{-1}{m}\right)\right)\left(1 + (-1)^\alpha\left(\frac{u}{3}\right)\right)\sum_{\nu|u}\left(\frac{-1}{\nu}\right) \ \textit{if}\ \ \alpha \geq 2,\ \ \beta = 0,\\ &= 2\left(1 + \left(\frac{-1}{m}\right)\right)\sum_{\nu|u}\left(\frac{-1}{\nu}\right)\ \textit{if}\ \ 2|\beta,\ \ \beta \geq 2,\ \ \alpha \geq 2,\\ &= 0\ \ \textit{if}\ \ \alpha = 1\ \ \textit{or}\ \ 2 \nmid \beta. \end{aligned}$$

Corollary 2. *Let* $f_1 = x^2 + 4\,y^2,\ f_2 = 4\,x^2 + 2\,xy + 11\,y^2,\ f_3 = 4\,x^2 - 2\,xy + 11\,y^2$, $n = 2^\alpha 43^\beta u,\ (u, 86) = 1$. *Then for* $\alpha \geq 1$, *we have*

$$\begin{aligned} r(n; f_1) = r(n; f_2) = r(n; f_3) &= \left(1 + \left(\frac{u}{43}\right)\right)\sum_{\nu|u}\left(\frac{-43}{\nu}\right)\ \ \textit{if}\ \ 2|\alpha,\\ &= 0\ \ \textit{if}\ \ 2 \nmid \alpha. \end{aligned}$$

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