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ABOUT ONE CONSTRUCTION OF STOCHASTIC INTEGRAL

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Abstract. We study the so called backward stochastic integral which in one specific case has been introduced by McKean (1969), but we consider a case when the integrand in addition depends on random and time variables. In particular, the class of integrable in such sense integrands is described, the relationships with both of Ito's and Stratanovich's integrals are established and the change of veriables rule (the analogous of Ito's formula) is given.

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1 Introduction. As it is known, in ordinary integration theory the requirement of measurability of integrand is essentially less restriction, than the condition of its integrability, which demands a certain bound condition of absolute value of the integrand. As for the Ito's Stochastic Integral $(I)\int_0^T f(s,\omega)\,dW_s(\omega)$, here, the situation is opposite in some sense: moreover, integrand $f(s,\omega)$ is a mesurable function of two variables, it should be an adapted process. On the one hand, it is clear that in many cases this is a natural condition, where the filtration $\Im_s^W := \sigma\{W_t : t \in [0,s]\}$ represents the evolution of the available information.

We define and study some properties of the backward stochastic integral where the integrand (in difference from the case considered in [1]) depends on random and time variables. Besides, even for step-processes, values of the natural sums of the integral depends on which endpoints of interval we will fix the value of integrand. For example (see, [2]), if partition $0 = t_0^n < t_1^n \cdots < t_n^n = T$ of the interval $[0,T]$ such that $\max_{0\le k\le n-1}\left|t_{k+1}^n - t_k^n\right| \to 0$ as $n\to\infty$, then for $\forall\alpha\in[0,1]$:

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left[(1-\alpha)\,W_{t_k^n}+\alpha W_{t_{k+1}^n}\right]\left(W_{t_{k+1}^n}-W_{t_k^n}\right)=\frac{1}{2}W_T^2+\left(\alpha-\frac{1}{2}\right)T.$$

It is easy to notice that the case $\alpha = 0$ corresponds to the construction of Ito's stochastic integral, whereas the case $\alpha = \frac{1}{2}$ corresponds to the construction of Stratanovich's stochastic integral $(S)\int_0^T f(s,\omega)\,dW_s(\omega)$. It is evident, that the dependence of this limit to the value of α is an outcome of the unboundence of the variance of Wiener process trajectories. We consider the analogous of the backward $(\alpha = 1)$ integrals $(B)\int_0^T f(W_t)\,dW_t$, $f \in C^1(R^1)$, which has been offered by McKean, but the integrand f in addition depends on t and ω. We consider the following situations: 1) f is dependent on t and ω and is a semimartingale; 2) $f(t,x) \in C^{1,1}$ and 3) $f = f(t,\eta_t,\omega)$, where η_t is a random process having the stochastic differential.

2 Main results

Proposition 1. *If adapted to the filtration $\Im_t^W$ continuous process $f(t)$ is a semimartingale with martingale part $M(t)$, then corresponding backward integral is represented as a sum of predictable square characteristics $\langle M, W\rangle$ and Ito's integral.*

Proof. Let $f(t,\omega) = f(0,\omega) + M(t,\omega) + A(t,\omega)$, where $M(t,\omega)$ is a local martingale with continuous paths and $A(t,\omega)$ is a process with bounded variation and continuous paths. Due to the definitions of predictable square characteristics and Ito's stochastic integral it is not difficult to see that:

$$(B)\int_0^T f(t,\omega)\,dW_t := \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(t_{k+1}^n,\omega\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1}\left\{f\left(t_{k+1}^n,\omega\right) - f\left(t_k^n,\omega\right)\right\}\left[W_{t_{k+1}^n} - W_{t_k^n}\right] + \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(t_k^n,\omega\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]$$

$$= \langle f, W\rangle_T + (I)\int_0^T f(t,\omega)\,dW_t = \langle M, W\rangle_T + (I)\int_0^T f(t,\omega)\,dW_t.$$

□

Corollary 1. *If the process $f(t)$ has bounded variation (i.e., $M(t) = 0$), then backward integral coincides with Ito's integral.*

Corollary 2. *If $f(t)$ is Ito's process with stochastic differential $df(T,\omega) = b(t,\omega)\,dW_t + a(t,\omega)\,dt$, then*

$$(B)\int_0^T f(t,\omega)\,dW_t = \int_0^T b(t,\omega)\,dt + (I)\int_0^T f(t,\omega)\,dW_t.$$

Due to the relations $f(t, W_t(\omega)) = g(t,\omega)$ and $f(t,\eta_t(\omega),\omega) = h(t,\omega)$, analogously of the Proposition 1, we have

Proposition 2. *The following realtions are true*

a) $$(B)\int_0^T f(t,W_t)dW_t = \langle f(\cdot\,,W), W\rangle_T + (I)\int_0^T f(t,W_t)dW_t,$$

b) $$(B)\int_0^T f(t,\eta_t,\omega)dW_t = \langle f(\cdot,\eta,\omega), W\rangle_T + (I)\int_0^T f(t,\eta_t,\omega)\,dW_t.$$

whenever the right hand side expressions are defined.

Corollary 3. *If $f(t,x) \in C^{1,2}$, using Ito's formula, we obtain*

$$(B)\int_0^T f(t,W_t)\,dW_t = \int_0^T f_x'(t,W_t)\,dt + (I)\int_0^T f(t,W_t)\,dW_t.$$

Corollary 4. *If for a.e. ω the function $f(\cdot,\cdot,\omega) \in C^{1,2}$, $df(t,x) = J(t,x)\,dt + H(t,x)\,dW_t$ and $d\eta_t = b_t dt + \sigma_t dW_t$, then by the Ito-Ventsel formula, we easily obtain*

$$(B)\int_0^T f(t,\eta_t,\omega)\,dW_t = \int_0^T \left[H(t,\eta_t) + f'_x(t,\eta_t)\,\sigma_t\right]dt + (I)\int_0^T f(t,\eta_t,\omega)\,dW_t.$$

Theorem 1. *If $f(t,x) \in C^{1,1}$, then the following relation is fulfilled*

$$(B)\int_0^T f(t,W_t)\,dW_t = \langle f(\cdot,W),W\rangle_T + (I)\int_0^T f(t,W_t)\,dW_t.$$

Proof. According to the mean value theorem, using the well-known properties of Wiener process, we can ascertain that:

$$(B)\int_0^T f(t,W_t)\,dW_t := \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(t_{k+1}^n, W_{t_{k+1}^n}\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1} f'_x\left(t_{k+1}^n, (1-\theta)\,W_{t_{k+1}^n} + \theta W_{t_k^n}\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]^2$$

$$+ \lim_{n\to\infty}\sum_{k=0}^{n-1} f'_t\left((1-\alpha)\,t_{k+1}^n + \alpha t_k^n, W_{t_k^n}\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]\left[t_{k+1}^n - t_k^n\right]$$

$$+ (I)\int_0^T f(t,W_t)\,dW_t = \int_0^T f'_x(t,W_t)\,dt + (I)\int_0^T f(t,W_t)\,dW_t.$$

□

Theorem 2. *If $f \in C^1(R^1)$, then we have*

$$(S)\int_0^T f(W_t)dW_t = \frac{1}{2}\int_0^T f'_x(W_t)dt + (I)\int_0^T f(W_t)dW_t.$$

Proof. It is not difficult to see that:

$$(S)\int_0^T f(W_t)\,dW_t := \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(W_{\frac{t_k^n + t_{k+1}^n}{2}}\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1}\left[f\left(W_{\frac{t_k^n + t_{k+1}^n}{2}}\right) - f\left(W_{t_k^n}\right)\right]\left[W_{t_{k+1}^n} - W_{t_k^n}\right] + \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(W_{t_k^n}\right)\left[W_{t_{k+1}^n} - W_{t_k^n}\right]$$

$$:= A_1 + A_2 + (I)\int_0^T f(W_t)\,dW_t.$$

Using now the mean value theorem, due to the well-known properties of Wiener process and Riemann's integral, on the one hand $A_1 = 0$ (taking into account $dW_t dt = 0$) and on the other hand (taking into account $(dW_t)^2 = dt$) we have

$$A_2 := \lim_{n\to\infty} \sum_{k=0}^{n-1} f'_x((1-\beta)W_{\frac{t_k^n+t_{k+1}^n}{2}} + \beta W_{t_k^n})[W_{\frac{t_k^n+t_{k+1}^n}{2}} - W_{t_k^n}]^2$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{k=0}^{n-1} f'_x(W_{\frac{t_k^n+t_{k+1}^n}{2}})(t_{k+1}^n - t_k^n) = \frac{1}{2}\int_0^T f'_x(W_t)dt.$$

□

Corollary 5. *If $f(t,x) \in C^{1,2}$, then due to Ito's formula, we easily conclude that the corresponding Ito's formula in Stratonovich mean coincides with total differential formula for two variables from classical analysis.*

According to the relation between Ito's and backward integrals we easily obtain the change of veriables rule for backward integral (the so called Ito's formula):

Proposition 3.

$$f(T,W_T) = f(0,W_0) + (I)\int_0^T f'_x(t,W_t)dW_t + \int_0^T \left[f'_t(t,W_t) + \frac{1}{2}f''_{xx}(t,W_t)\right]dt$$

$$= f(0,W_0) + (B)\int_0^T f'_x(t,W_t)dW_t - \int_0^T f''_{xx}(t,W_t)dt + \int_0^T \left[f'_t(t,W_t) + \frac{1}{2}f''_{xx}(t,W_t)\right]dt$$

$$= f(0,W_0) + (B)\int_0^T f'_x(t,W_t)dW_t + \int_0^T \left[f'_t(t,W_t) - \frac{1}{2}f''_{xx}(t,W_t)\right]dt.$$

It should be noted that one can receive this relation by straight lines using a method similar to that which was used in the proof of the usual Ito's formula but here instead of a usual formula of Taylor it is necessary to use the so called backward formula of Taylor.

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