ON THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL MITCHISON NONLINEAR MODEL

Besik Tabatadze

Abstract. In the note for the construction of numerical solution of two-dimensional Mitchison nonlinear partial differential system the variable directions difference scheme and the difference scheme corresponding to the average method are used. Practical realization of those algorithms is fulfilled and comparative analysis of the obtained results are carried out. Numerical experiments are in accordance with theoretical findings. On basis of experiments the corresponding tables of data is given.

Keywords and phrases: Nonlinear partial differential equations, initial-boundary value problem, finite difference scheme.


By the nonlinear partial differential equations a lot of natural processes are described. Among them, is one of the important mathematical model, that describes vein formation in the leaves of plants. This model was proposed by J. Michison [1].

Investigations for one-dimensional analog of Michison system are carried out in [2].

Beginning from the basic works the methods of constructing of effective algorithms for the numerical solution of the multi-dimensional problems of the mathematical physics and the class of problems solvable with the help of these algorithms were essentially extended [3] - [7]. Those algorithms mainly belong to the methods of splitting-up or sum approximation. Some schemes of the variable directions are constructed and studied in the following work too [8].

Some questions of construction and investigation of the schemes of variable directions and the average model of sum approximation as well as difference schemes for one-dimensional case for the type Michison systems are discussed in the papers [9] - [15].

Let us take special form of function \( f \) and consider the following two-dimensional system:

\[
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left( V_1 \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( V_2 \frac{\partial U}{\partial y} \right),
\]

\[
\frac{\partial V_1}{\partial t} = -V_1 + g_1 \left( V_1 \frac{\partial U}{\partial x} \right), \quad \frac{\partial V_2}{\partial t} = -V_2 + g_2 \left( V_2 \frac{\partial U}{\partial y} \right)
\]

(1)

with the initial

\[
U(x, y, 0) = U_0(x, y), \quad V_1(x, y, 0) = V_{10}(x, y), \quad V_2(x, y, 0) = V_{20}(x, y)
\]

(2)
and boundary conditions

$$U(x, 0, t) = U(0, y, t) = U(x, 1, t) = U(1, y, t) = 0, \quad t \in [0, T], \quad x, y \in [0, 1]. \quad (3)$$

Here $g_\alpha, U_0, V_{\alpha 0} \quad \alpha = 1, 2$ are given sufficiently smooth functions, such that:

$$V_{\alpha 0} \geq \delta_0, \quad \delta_0 = const > 0, \quad (x, y) \in \bar{\Omega},$$

$$g_0 \leq g_\alpha(\xi_\alpha) \leq G_0, \quad |g_\alpha(\xi_\alpha)| \leq G_1, \quad \xi_\alpha \in R, \quad \alpha = 1, 2, \quad (4)$$

where $\delta_0, g_0, G_0, G_1$ and $T$ are some positive constants.

Later we shall follow known notations for the construction of the grid on the domain $\bar{Q}$:

$$\bar{\omega}_h = \{(x_i, y_j) = (ih, jh)\},$$

$$\bar{\omega}_{1h} = \{(x_i, y_j) = ((i - 1/2)h, jh)\},$$

$$\bar{\omega}_{2h} = \{(x_i, y_j) = (ih, (j - 1/2)h)\},$$

$$i, j = 0, ..., M, Mh = 1,$$

$$\omega_h = \Omega \cap \bar{\omega}_h, \gamma_h = \bar{\omega}_h/\omega_h, \bar{\omega}_h = \omega_h \cup \gamma_h,$$

$$\omega_r = \{t_k = k\tau, k = 0, ..., N, N\tau = T\}. \quad (5)$$

Following the known notations [7], let us correspond to problem (1) the following difference scheme of variable directions [10]:

$$u_{1t} = (\hat{v}_1 \hat{u}_{1\xi})_x + (v_2 u_{2g})_y, \quad u_{2t} = (\hat{v}_1 \hat{u}_{1\xi})_x + (\hat{v}_2 u_{2g})_y,$$

$$v_{1t} = -\hat{v}_1 + g_1(v_1 u_{1\xi}), \quad v_{2t} = -\hat{v}_2 + g_2(v_2 u_{2g}),$$

$$u_1(x, y, 0) = U_0(x, y), \quad u_2(x, y, 0) = U_0(x, y), \quad (x, y) \in \bar{\omega}_h,$$

$$v_1(x, y, 0) = V_{10}, \quad v_2(x, y, 0) = V_{20}, \quad (x, y) \in \bar{\omega}_{2h},$$

$$u_1(x, y, t) = u_2(x, y, t) = 0, \quad (x, y, t) \in \gamma_h \times \omega_r. \quad (6)$$

For problem (1) let us also consider the following difference scheme corresponding to the average method [9]:

$$u_{1t} = (\hat{v}_1 \hat{u}_{1\xi})_x, \quad u_{2t} = (\hat{v}_2 u_{2g})_y,$$

$$v_{1t} = -\hat{v}_1 + g_1(v_1 u_{1\xi}), \quad v_{2t} = -\hat{v}_2 + g_2(v_2 u_{2g}),$$

$$u_1(x, y, 0) = U_0(x, y), \quad u_2(x, y, 0) = U_0(x, y), \quad (x, y) \in \bar{\omega}_h, \quad (7)$$
\[ v_1(x, y, 0) = V_{10}, \quad v_2(x, y, 0) = V_{20}, \quad (x, y) \in \bar{\omega}_h, \]
\[ u_1(x, y, t) = u_2(x, y, t) = 0, \quad (x, y, t) \in \gamma_h \times \omega, \]
\[ u = \eta_1 u_1 + \eta_2 u_2, \quad \eta_1 \eta_2 > 0, \quad \eta_1 + \eta_2 = 1, \]
\[ u_1 = u, \quad u_2 = u. \]

Using algorithms, proposed in (6) and (7) let us carry out comparative analysis of the numerical results for schemes above.

Let us take the right side of the corresponding nonhomogeneous system (1) so that the solution of problem (1), (2) is:
\[
U(x, y, t) = xy(1-x)(1-y)(1+t),
\]
\[
V_1(x, y, t) = 1 + xy(1-x)(1-y)(1+t + t^2),
\]
\[
V_2(x, y, t) = 1 + xy(1-x)(1-y)(1+t + t^3).
\]

CPU time and errors for variable directions difference scheme (6) are given in the table 1 and the CPU time and errors for the scheme (7) are given in the table 2.

**Table 1:** CPU time and error for solution $U$, $V_1$, $V_2$ applying variable directions difference scheme (6).

<table>
<thead>
<tr>
<th>t</th>
<th>CPU time</th>
<th>Error $U$</th>
<th>Error $V_1$</th>
<th>Error $V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.074</td>
<td>0.00013912790131447</td>
<td>0.0000712766408961</td>
<td>0.0002916084998672</td>
</tr>
<tr>
<td>0.4</td>
<td>0.148</td>
<td>0.00022425859907783</td>
<td>0.0001730244454379</td>
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<tr>
<td>0.6</td>
<td>0.224</td>
<td>0.00031286373416026</td>
<td>0.0004804529821700</td>
<td>0.0017715471240609</td>
</tr>
<tr>
<td>0.8</td>
<td>0.301</td>
<td>0.00040788793632886</td>
<td>0.0009668298990784</td>
<td>0.0028758192640277</td>
</tr>
<tr>
<td>1.0</td>
<td>0.378</td>
<td>0.00051151056363487</td>
<td>0.0016425091499817</td>
<td>0.0041715052893787</td>
</tr>
</tbody>
</table>

**Table 2:** CPU time and error for solution $U$, $V_1$, $V_2$ applying variable directions difference scheme (7).

<table>
<thead>
<tr>
<th>t</th>
<th>CPU time</th>
<th>Error $U$</th>
<th>Error $V_1$</th>
<th>Error $V_2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.00006973950435170</td>
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<tr>
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<tr>
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<td>0.00007978157763566</td>
</tr>
<tr>
<td>1.0</td>
<td>0.369</td>
<td>0.00009205402490850</td>
<td>0.00011818090303972</td>
<td>0.00010625023389577</td>
</tr>
</tbody>
</table>

The approximation error for variable direction difference scheme (6) is smaller compared with scheme (7). However, CPU time is better for scheme (7) than scheme (6). We have experimented a number of other experiments and observed the same situations. In all cases results of numerical experiments are in accordance with theoretical findings.
REFERENCES


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