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MATHEMATICAL ANALYSIS OF THE EARLY EXERCISE BOUNDARY FOR THE AMERICAN PUT OPTION

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Abstract. We generalize in this paper the well known continuity property of the early exercise boundary for the American put option in the classical geometric Brownian motion model to the case of a diffusion model with level-dependent volatility. The proof of the continuity in the classical case given in Karatzas, Shreve [1, Chapter 2] heavily relies on the differentiability of the value function in the neighborhood of the early exercise boundary. On the contrary, our probabilistic proof does not require any type of differentiability of the value function and is based on the appropriate application of the general Dynamic Programming principle.

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1 Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider a standard Wiener process $W = (W_t, \mathcal{F}_t)$, $0 \le t \le T$, on it. We assume that the time horizon T is finite. We will consider on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $0 \le t \le T$, the financial market with two assets (B_t, X_t), $0 \le t \le T$, where B_t is the value at time t of the unit bank account and X_t is the stock value at time t. The dynamics of these values obeys the following ordinary and stochastic differential equations

$$dB_t = r \cdot B_r \, dt, \quad B_0 = 1, \ \ 0 \le t \le T, \tag{1.1}$$

$$dX_t = r \cdot X_t \, dt + \sigma(X_t) \cdot X_t \, dW_t, \ \ 0 \le t \le T, \ \ X_0 > 0, \tag{1.2}$$

where $r > 0$ is the bank interest rate and $\sigma(x)$, $x \ge 0$, is called the local volatility function and satisfies the following conditions

$$0 < \underline{\sigma} \le \sigma(x) \le \overline{\sigma}, \ \ |x \cdot \sigma(x) - y \cdot \sigma(y)| \le c|x - y|. \tag{1.3}$$

The last condition guarantees the existence and the uniqueness of strong solution of the stochastic differential equation (1.2).

We will need the same stochastic differential equation started at arbitrary time instant s and from arbitrary state x, $x \ge 0$,

$$dX_u(s,x) = r \cdot X_u(s,x) \, du + \sigma(X_u(s,x)) \cdot X_u(s,x) \, dW_u, \ \ s \le u \le T, \ \ X_s(s,x) = x. \tag{1.4}$$

Let us consider the American put option on the stock with value X_t at time t and with a payoff function

$$g(x) = (K - x)^+. \tag{1.5}$$

Define the value function of the American put option

$$v(s,x)=\sup_{s\le\tau\le T} E\Big(e^{-r(\tau-s)}\cdot(K-X_\tau(s,x))^+\Big),\ \ 0\le s\le T,\ \ s\ge 0. \tag{1.6}$$

We remind that according to Theorem 5.8, Chapter 2 of Karatzas, Shreve [1] an optimal exercise time for the buyer of the option is the following

$$\tau_s^*=\inf\Big\{u\ge s:\ v(u,X_u(s,x))=(K-X_u(s,x))^+\Big\}\wedge T,\ \ 0\le s\le T, \tag{1.7}$$

and the famous Dynamic Programming equation takes the following form

$$v(s,x)=E\Big[e^{-r(\tau_s^*\wedge t-s)}\cdot v\big(\tau_s^*\wedge t, X_{\tau_s^*\wedge t}(s,x)\big)\Big],\ \ 0\le s\le T. \tag{1.8}$$

Now let us introduce the so called continuation or no-exercise domain of the American put option

$$C=\Big\{(s,x):\ \ 0\le s<T,\ \ x>0,\ \ v(s,x)>(K-x)^+\Big\} \tag{1.9}$$

and its early exercise boundary $b(s)$, $0\le s\le T$,

$$b(s)=\inf\big\{x\ge 0:\ \ v(s,x)>(K-x)^+\big\},\ \ 0\le s\le T, \tag{1.10}$$

where

$$b(s)\le K. \tag{1.11}$$

It is known (see, for example, Babilua, Bokuchava, Dochviri and Shashiashvili [2]) that $b(s)$, $0\le s\le T$, is a nondecreasing function of time. The objective of this paper is to prove the continuity of this early exercise boundary. We should note here that for the particular case of geometric Brownian motion such a proof relies on the differentiability properties of the value function $v(s,x)$ in the vicinity of the function $b(s)$ and especially on the so called smooth fit property. Our probabilistic proof does not use any type of differentiability of $v(s,x)$ and is based essentially on the general Dynamic Programming principle plus presently well known monotonicity in volatility property of the European as well as American options value functions with convex payoffs. The latter property was originally established by Ekstrom [3].

2 Comparison of the value functions and the proof of the continuity of the early exercise boundary. Let us introduce an artificial American put option problem will time horizon t, $0\le t\le T$, and with the strike price $b(t)$ at time t for the classical geometric Brownian motion model

$$\begin{gathered} d\overline{X}_u(s,x)=r\cdot\overline{X}_u(s,x)\,du+\overline{\sigma}\cdot\overline{X}_u(s,x)\,dW_u,\ \ 0\le s\le u\le t,\\ \overline{X}_s(s,x)=x,\ \ x\ge 0, \end{gathered} \tag{2.1}$$

Define now the value function of the latter American put option

$$\overline{v}(s,x)=\sup_{s\le\tau\le t} E\big[e^{-r(\tau-s)}\big(b(t)-\overline{X}_\tau(s,x)\big)^+\big],\ \ 0\le s\le t,\ \ x\ge 0. \tag{2.2}$$

The following proposition is the key to our probabilistic approach.

Theorem 1. *The following estimate is valid for the original American put value function* (1.6) *through the artificial value function* (2.2)

$$v(s,x) \le \overline{v}(s,x) + (K - b(t)), \quad 0 \le s \le t, \quad x \ge 0. \tag{2.3}$$

Proof. Consider the Dynamic Programming equation (1.8) and separate the right-hand side according to two events $(\tau_s^* > t)$ and $(\tau_s^* \le t)$, we shall get

$$\begin{aligned} v(s,x) = E\big[&e^{-r(\tau_s^*\wedge t-s)}\big(K - X_{\tau_s^*\wedge t}(s,x)\big)^+ \cdot I_{(\tau_s^*\le t)}\big] \\ &+ E\big[e^{-r(t-s)}v(t, X_t(s,x)) \cdot I_{(\tau_s^*>t)}\big]. \end{aligned} \tag{2.4}$$

We have

$$\big(K - X_{\tau_s^*\wedge t}(s,x)\big)^+ \le (K - b(t)) + \big(b(t) - X_{\tau_s^*\wedge t}(s,x)\big)^+.$$

On the other hand in case of the event $(\tau_s^* > t)$ we have $X_t(s,x) > b(t)$ and by the decreasing character of the value function $v(t,x)$ in x (see Babilua, Bokuchava, Dochviri, Shashiashvili [2]) we get

$$v(t, X_t(s,x)) \le v(t, b(t)) = (K - b(t)),$$

hence from the latter inequalities and equality (2.4) we obtain

$$v(s,x) \le E\big[e^{-r(\tau_s^*\wedge t-s)}\big(b(t) - X_{\tau_s^*\wedge t}(s,x)\big)^+\big] + (K - b(t)). \tag{2.5}$$

Finally, let us apply the monotonicity in volatility property (Ekstrom [3]) for the American put options to the latter inequality (2.5) and we get estimate (2.3). □

Now, we are ready to establish the basic result of this paper.

Theorem 2. *The early exercise boundary* $b(t)$, $0 \le t \le T$, *of the American put option* (1.6) *is continuous on the time interval* $[0,T]$.

Proof. We know that $b(t)$, $0 \le t \le T$, is a nondecreasing function, hence we get

$$b(t-) \le b(t), \quad 0 < t \le T. \tag{2.6}$$

Consider now the artificial American put option problem (2.2) on the time interval $[0,t]$ with strike price $b(t)$ and denote by $\overline{b}(s)$, $0 \le s \le t$, its early exercise boundary. It is a classical fact (see for example, Kim [4]) that $\overline{b}(s)$, $0 \le s \le t$, is a nondecreasing function and, moreover,

$$\overline{b}(t-) = b(t). \tag{2.7}$$

Put $x = \overline{b}(s)$ in the estimate (2.3), we shall have

$$v(s, \overline{b}(s)) \le \overline{v}(s, \overline{b}(s)) + (K - b(t)), \tag{2.8}$$

otherwise

$$(K - \overline{b}(s))^+ \le v(s, \overline{b}(s)) \le b(t) - \overline{b}(s) + K - b(t) = K - \overline{b}(s), \tag{2.9}$$

which implies the equality

$$v(s,\bar{b}(s)) = (K - \bar{b}(s))^+ \tag{2.10}$$

and therefore the majorization

$$\bar{b}(s) \le b(s), \ \ 0 \le s < t. \tag{2.11}$$

Let us pass to limit $s \uparrow t$ in the latter inequality, we shall have

$$\bar{b}(t-) \le b(t-) \le b(t). \tag{2.12}$$

But from equality (2.7) we have $\bar{b}(t-) = b(t)$ and hence we come to the desired left-continuity of the boundary $b(t)$ of the original American put option problem (1.6)

$$b(t-) = b(t), \ \ 0 < t \le T. \tag{2.13}$$

It follows from the definition (1.10) of the early exercise boundary $b(t)$, that it is the upper semicontinuous function and thus $b(t) \ge b(t+)$, $0 \le t < T$, but the reverse inequality is true as the function $b(t)$ is nondecreasing and hence the equality $b(t) = b(t+)$, $0 \le t < T$. Ultimately we get the required result

$$b(t-) = b(t) = b(t+), \ \ 0 \le t \le T. \tag{2.14}$$

□

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