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ON A NUMERICAL ALGORITHM FOR A TIMOSHENKO TYPE NONLINEAR BEAM EQUATION *

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Abstract. An initial boundary value problem is considered for the J. Ball dynamic beam equation. The solution is found by means of an algorithm, the constituent parts of which are the Galerkin method, an implicit symmetric difference scheme and Jacobi iterative method. The approximation error is estimated.

Keywords and phrases: Nonlinear beam equation, Galerkin method, implicit symmetric difference scheme, Jacobi iterative method, error estimate.

AMS subject classification (2010): 65M60, 65M06, 65Q10, 65M15.

1 Statement of the problem.

Let us consider the nonlinear equation

$$\begin{gathered} u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t)- \\ \left(\beta+\rho\int\limits_0^L u_x^2(x,t)dx\right)u_{xx}(x,t)- \\ \sigma\left(\int\limits_0^L u_x(x,t)u_{xt}(x,t)dx\right)u_{xx}(x,t)=0,\quad 0<x<L,\quad 0<t\le T, \end{gathered} \tag{1}$$

with the initial boundary conditions

$$\begin{gathered} u(x,0)=u^0(x),\qquad u_t(x,0)=u^1(x), \\ u(0,t)=u(L,t)=0,\quad u_{xx}(0,t)=u_{xx}(L,t)=0. \end{gathered} \tag{2}$$

Here $\alpha,\gamma,\rho,\sigma,\beta$ and δ are given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$ are given functions such that $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$. It will be assumed that the inequality $|\delta|<\gamma\left(\frac{\pi}{L}\right)^4$ is fulfilled when $\delta<0$, and $\alpha\left(\frac{\pi}{L}\right)^2>|\beta|$ holds when $\beta<0$. We suppose that there exits a solution $u(x,t)\in W_2^2\left((0,L)\times(0,T)\right)$ of problem (1), (2).

The equation (1) obtained by J.Ball [1] using the Timoshenko theory describes the vibration of a beam. The problem of construction of an approximate solution for this equation is dealt with in [2], [3]. In [1], the existence of a global for (1), (2) is shown.

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2 Algorithm and its error

2.1 Galerkin method. We write an approximate solution of problem (1), (2) in the form

$u_n(x,t)=\sum_{i=1}^{n} u_{ni}(t)\sin\frac{i\pi x}{L}$, where the coefficients $u_{ni}(t)$ will be found by the Galerkin method from the system of ordinary differential equations

$$u''_{ni}(t)+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)u'_{ni}(t)+\left\{\alpha\left(\frac{i\pi}{L}\right)^4+\left(\frac{i\pi}{L}\right)^2\times\right.$$
$$\left.\left[\beta+\rho\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}^2(t)+\sigma\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}(t)u'_{nj}(t)\right]\right\}u_{ni}(t)=0, \tag{3}$$
$$i=1,2,\cdots,n,\quad 0<t\le T,$$

with the initial conditions

$$u_{ni}(0)=a_i^0,\quad u'_{ni}(0)=a_i^1,\quad i=1,2,\cdots,n, \tag{4}$$

where

$$a_i^p=\frac{2}{L}\int_0^L u^p(x)\sin\frac{i\pi}{L}xdx,\quad p=0,1,\quad i=1,2,\cdots,n.$$

The error of a Galerkin method is estimated in the article [4].

2.2 Difference scheme. To solve problem (3), (4) we apply the difference method. On the time interval $[0,T]$ we introduce a net with step $\tau=\frac{T}{M}$ and nodes $t_m=m\tau$, $m=0,1,2,\cdots,M$.

On the m-th layer, i.e. for $t=t_m$, the approximate value of $u_{ni}(t_m)$ is denoted by u_{ni}^m. We use an implicit symmetric difference scheme

$$\frac{u_{ni}^{m+1}-2u_{ni}^m+u_{ni}^{m-1}}{\tau^2}+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)\frac{u_{ni}^{m+1}-u_{ni}^{m-1}}{2\tau}+$$
$$\left\{\alpha\left(\frac{i\pi}{L}\right)^4+\left(\frac{i\pi}{L}\right)^2\left(\beta+\rho\frac{L}{2}\sum_{j=1}^{n}\frac{1}{2}\left(\frac{j\pi}{L}\right)^2\times\right.\right.$$
$$\left[\left(\frac{u_{nj}^{m+1}+u_{nj}^m}{2}\right)^2+\left(\frac{u_{nj}^m+u_{nj}^{m-1}}{2}\right)^2\right]+ \tag{5}$$
$$\left.\left.\sigma\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2\frac{u_{nj}^{m+1}+2u_{nj}^m+u_{nj}^{m-1}}{4}\times\frac{u_{nj}^{m+1}-u_{nj}^{m-1}}{2\tau}\right)\right\}\times$$
$$\frac{u_{ni}^{m+1}+2u_{ni}^m+u_{ni}^{m-1}}{4}=0,\qquad m=1,2,\cdots,M-1,\quad i=1,2,\cdots,n;$$

with the conditions

$$u_{ni}^0 = a_i^0, \quad u_{ni}^1 = a_i^0 + \tau a_i^1 + \frac{\tau^2}{2} a_i^2, \quad i = 1, 2, \cdots, n, \tag{6}$$

$$a_i^2 = -\left\{ \left(\delta + \gamma \left(\frac{i\pi}{L} \right)^4 \right) a_i^1 + \right.$$
$$\left. \left[\alpha \left(\frac{i\pi}{L} \right)^4 + \left(\frac{i\pi}{L} \right)^2 \left(\beta + \rho \frac{L}{2} \sum_{j=1}^{n} \left(\frac{j\pi}{L} \right)^2 a_j^{0\,2} + \sigma \frac{L}{2} \sum_{j=1}^{n} \left(\frac{j\pi}{L} \right)^2 a_j^0 a_j^1 \right) \right] a_i^0 \right\}.$$

Let's determine the scheme (5), (6) error for function $u_{ni}(t)$ in a point $t = t_m$ by a difference

$$\Delta u_{ni}^m = u_{ni}^m - u_{ni}(t_m), \quad m = 1, 2, \cdots, M.$$

At condition of a sufficient smallness of a step τ which registers by means of some inequalities, takes place an assessment of accuracy of the difference scheme

$$\sum_{i=1}^{n} |\Delta u_{ni}^m| \leq \sum_{i=1}^{n} \left| \frac{\Delta u_{ni}^m - \Delta u_{ni}^{m-1}}{\tau} \right| + \sum_{i=1}^{n} \left| \frac{\Delta u_{ni}^m + \Delta u_{ni}^{m-1}}{2} \right| \leq$$
$$c_1 \max_{1 \leq s \leq m-1} \sum_{i=1}^{n} |\Psi_{ni}^s|, \quad m = 2, 3, \cdots, M, \quad c_1 = (1 + \tau p_1)(1 + \tau p_2),$$

where $p_j > 0$ values do not depend on τ and are expressed by means of parameters of a task (1), (2), and Ψ_{ni}^s is the error of approximation of the scheme.

2.3 Iterative method. System (5), (6) will be solved layer-by-layer. Assuming that the solution has been already been obtained on the $(m-1)$-th and m-th layer to find it on the $(m+1)$-th layer we use the Jacobi iterative method. For the sake of simplicity, the error of the final iterative approximation on the $(m-1)$-th and m-th layers will be neglected.

The system of equations adequate (appropriate) iterative method comes down to the set of the following equations by means of a Cardano formula (see [5])

$$\begin{aligned} & u_{ni,k+1}^{m+1} = \varphi_{ni} \left(u_{n1,k}^{m+1}, u_{n2,k}^{m+1}, \cdots, u_{nn,k}^{m+1} \right), \\ & i = 1, 2, \cdots, n, \end{aligned} \tag{7}$$

$$m = 1, 2, \cdots, M-1, \quad k = 0, 1, \cdots, \quad i = 1, 2, \cdots, n,$$

where $u_{ni,k+p}^{m+1}$ denotes the $(k+p)$-th iterative approximation of u_{ni}^{m+1}, $i = 1, 2, \cdots, n$, $p = 0, 1$, u_{ni}^m and u_{ni}^{m-1} are the known values.

It is proved that if the step satisfies some conditions of smallness which are written down by means of inequalities, then the iteration process (7) converges and for the error the following inequalities are true:

$$\left(\sum_{i=1}^{n}\frac{1}{2}\left(\frac{i\pi}{L}\right)^{2p}\left(u_{ni,k}^{m}-u_{ni}^{m}\right)^{2}\right)^{1/2}\leq c_2\left(\frac{L}{\pi}\right)^{1-p}\frac{q^k}{1-q}\times$$
$$\left(\sum_{i=1}^{n}\frac{1}{2}\left(\frac{i\pi}{L}\right)^{2p}\left(u_{ni,1}^{m}-u_{ni,0}^{m}\right)^{2}\right)^{1/2},$$

$$p=0,1,\quad k=1,2,\cdots,\quad m=1,2,\cdots,M,$$

where $c_2>0$ and $0<q<1$ are values which do not depend on τ and k, and are expressed by the parameters of a task (1), (2).

Remark 1. The question of estimation of the error for (5)-(6) symmetric difference scheme is new and is not published yet. Investigation of accuracy of difference scheme for dynamic beam involves: a) additional inequalities for estimation; b) the system of equations for the error; c) the system of equations, written in a matrix form; d).estimation of the error of difference scheme.

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