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ON ONE METHOD OF APPROXIMATE SOLUTION OF KIRCHHOFF TYPE STATIC BEAM NONLINEAR INTEGRO-DIFFERENTIAL EQUATION *

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Abstract. The paper deals with a boundary value problem for the Kirchhoff type static beam nonlinear integro-differential equation. The problem is reduced by Green function to an integral equation which is solved using the Picard iteration method. The convergence of the iteration process is established and numerical realization is obtained.

Keywords and phrases: Kirchhoff type static beam equation, Green function, integral equation, Picard iteration method, numerical realization.

AMS subject classification (2010): 34B27, 65L10, 65R20.

1 Statement of the problem.

Let us consider the nonlinear beam equation

$$u''''(x) - m\left(\int_0^l u'^2(x)dx\right)u''(x) = f(x, u(x), u'(x)), \quad 0 < x < l, \tag{1}$$

with the conditions

$$u(0) = u(l) = 0, \quad u''(0) = u''(l) = 0. \tag{2}$$

Here $u = u(x)$ is the displacement function of length l of the beam subjected to the action of a force given by the function $f(x, u(x), u'(x))$, the function $m(z)$,

$$m(z) \geq \alpha > 0, \quad 0 \leq z < \infty \tag{3}$$

describes the type of a relation between stress and strain. Namely, if the function $m(z)$ is linear, this means that this relation is consistent with Hooke's linear law, while otherwise we deal with material nonlinearites. The questions of the solvability of problem (1), (2) is studied in [1], while the problem of construction of numerical algorithms and estimation of their accuracy is investigated in [2]-[5]. In the present paper, in order to obtain an approximate solution of problem (1), (2) an approach is used, which differs from those applied in the above-mentioned references. It consists in reducing the problem (1), (2) by means of Green's function to a nonlinear integral equation, to solve whith we use the iterative process. The condition for the convergence of the method is estabilished and numerical realization is obtained.

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2 The method.

We will need the Green function for the problem

$$
\begin{aligned}
&v''''(x) - av''(x) = \psi(x),\\
&0 < x < l, \quad a = const > 0,\\
&v(0) = v(l) = 0, \quad v''(0) = v''(l) = 0.
\end{aligned}
\tag{4}
$$

In order to obtain this function, we split problem (4) into two problems, which by Green function and after integral representations some transformation gives:

$$
\begin{aligned}
&v(x) = \frac{1}{a}\times\\
&\left(\int_0^x \left(\frac{1}{l}(x-l)\xi + \frac{1}{\sqrt{a}\sin h(\sqrt{a}l)}\sin h(\sqrt{a}(x-l))\sin h(\sqrt{a}\xi)\right)\psi(\xi)d\xi + \right.\\
&\left.\int_x^l \left(\frac{1}{l}x(\xi-l) + \frac{1}{\sqrt{a}\sin h(\sqrt{a}l)}\sin h(\sqrt{a}(\xi-l))\sin h(\sqrt{a}x)\right)\psi(\xi)d\xi\right).
\end{aligned}
\tag{5}
$$

The application of (4), (5) to problem (1), (2) makes it possible to replace the latter problem by the integral equation

$$
u(x) = \int_0^l G(x,\xi) f(\xi, u(\xi), u'(\xi))d\xi, \quad 0 < x < l, \tag{6}
$$

where

$$
G(x,\xi) = \frac{1}{\tau}\begin{cases}
\dfrac{1}{l}(l-x)\xi + \dfrac{1}{\sqrt{\tau}\sin h(\sqrt{\tau}l)}\sin h(\sqrt{\tau}(x-l))\sin h(\sqrt{\tau}\xi), \\
\qquad\qquad 0 < \xi \le x < l,\\
\dfrac{1}{l}x(l-\xi) + \dfrac{1}{\sqrt{\tau}\sin h(\sqrt{\tau}l)}\sin h(\sqrt{\tau}(\xi-l))\sin h(\sqrt{\tau}x), \\
\qquad\qquad 0 < x \le \xi < l,
\end{cases}
$$

$$
\tau = m\left(\int_0^l u'^2(x)dx\right).
$$

The equation (6) is solved by the method of the Picard iterations. After choosing the function $u_0(x), 0 \le x \le l$, which together with its second derivative vanish for the $x = 0$ an $x = l$, we find subsequent approximations by the formula

$$
u_{k+1}(x) = \int_0^l G_k(x,\xi) f(\xi, u_k(\xi), u_k'(\xi))d\xi, \quad 0 < x < l, \quad k = 0, 1, \cdots \tag{7}
$$

where

$$G_k(x,\xi)=\frac{1}{\tau_k}\begin{cases}\frac{1}{l}(l-x)\xi+\frac{1}{\sqrt{\tau_k}\sin h(\sqrt{\tau_k}l)}\sin h(\sqrt{\tau_k}(x-l))\sin h(\sqrt{\tau_k}\xi), \\ \qquad\qquad 0<\xi\le x<l, \\ \frac{1}{l}x(l-\xi)+\frac{1}{\sqrt{\tau_k}\sin h(\sqrt{\tau_k}l)}\sin h(\sqrt{\tau_k}(\xi-l))\sin h(\sqrt{\tau_k}x), \\ \qquad\qquad 0<x\le\xi<l,\end{cases}$$

$$\tau_k=m\left(\int\limits_0^l u_k'^2(x)dx\right).$$

$k=0,1,\cdots$ and $u_k(x)$ is the k-th approximation of the solution of equation (6).

3 The numerical realization. For approximate solution of the boundary value problem (1), (2) some programs in algorithm language Maple are composed and many numerical experiments are carried out. The obtained results are good enough. The algorithm has been approved tests and the results of recounts are represented in graphics. The algorithm is approved in the following two tasks on the test:

Test - 1. The given parameters $m_0=m_1=1,\ l=1$, the right hand side $f(x,u(x),u'(x))=1+(u(x))^2+(u'(x))^2$.

Test - 2. The given parameters $m_0=m_1=1,\ l=1$, the right hand side $f(x,u(x),u'(x))=\cos(x)\,(1+(u(x))^2+(u'(x))^2)$.

For both tests five iterations have been performed. For calculation of integrals generalized trapezoidal quadratic formula has been used with divided [0,1] segment by n=20 parts. The results of numerical computations at the point x=0.5 (see table 1) and the graphics of solution (see Fig. 1_a and Fig. 1_b) are presented.

test N | Number of iteration	1	2	3	4	5
test 1	0.2382	0.2150	0.2153	0.2156	0.2156
test 2	0.2048	0.1892	0.1894	0.1895	0.1895

Table 1.

Remark 1. In figures the graph of the first approximation is green. Starting from the second approximation the graphs coincide with each other. We have a rapidly converging iterative method. Other test problems can be seen in [5].

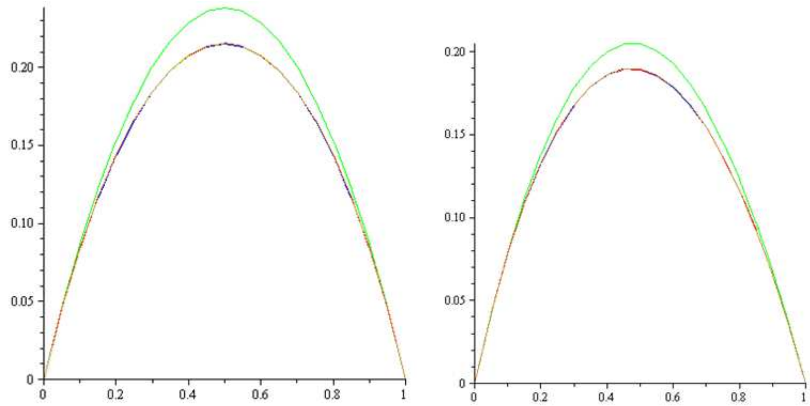

Figure 1: a, b

R E F E R E N C E S

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