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A SYSTEM OF FORWARD-BACKWARD SDES RELATED TO UTILITY MAXIMIZATION PROBLEM

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Abstract. The wealth maximization problem for utility function defined on the whole real line with random liability is considered. For the solution of this problem the system of Forward Backward Stochastic Differential Equations is derived.

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We consider a financial market model, where the dynamics of asset prices is described by the continuous semimartingale S defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $F = (F_t, t \in [0,T])$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and $T < \infty$. The bond is assumed to be a constant.

Let $U = U(x) : \mathbb{R} \to \mathbb{R}$ be a utility function such that U is continuously differentiable, increasing, strictly concave and $U'(\infty) = 0, \quad U'(-\infty) = \infty$.

We also assume that U satisfies the condition of reasonable asymptotic elasticity (see [4] for details), i.e.

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \quad \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{1}$$

Let V be the convex conjugate of U, i.e., $V(y) = \sup_x (U(x) - xy),\ y > 0$.

Denote by $\mathcal{M}^e$ (resp. $\mathcal{M}^a$) the set of probability measures Q equivalent (resp. absolutely continuous) to P such that S is a local martingale under Q. Following [4] and [2] we denote by $\mathcal{M}^e_V$ (resp. $\mathcal{M}^a_V$) the set of martingale measures $Q \in \mathcal{M}^e$ (resp. $Q \in \mathcal{M}^a$) such that $EV(\frac{dQ}{dP}) < \infty$ and throughout the paper assume that

$$\mathcal{M}^e_V \neq \emptyset. \tag{2}$$

The continuity of S and the existence of an equivalent martingale measure imply that S admits the decomposition

$$S_t = M_t + \int_0^t \lambda_s \, d\langle M\rangle_s, \quad \int_0^t \lambda_s^2 \, d\langle M\rangle_s < \infty, \ \ t \in [0,T], \ \ P-a.s.,$$

where M is a continuous local martingale and λ is a predictable process.
The wealth process is defined as a stochastic integral $X^\pi_t = x + (\pi \cdot S)_t, t \leq T$.

We consider the utility maximization problem of terminal wealth with random liability H

$$u_H(x) = \max_{\pi \in \Pi_x} EU\left(x + \int_0^T \pi_s dS_s + H\right), \tag{3}$$

where H is a bounded F_T-measurable random variable and Π_x is the class of predictable S-integrable processes π such that $\pi \cdot S$ is a Q-supermartingale for all $Q \in \mathcal{M}_V^a$.

The dual problem to (3) is

$$v_H(y) = \inf_{Q \in \mathcal{M}_V^e} E[V(yZ_T^Q) + yZ_T^Q H], \tag{4}$$

where $Z_t^Q = dQ_t/dP_t$ is the density process of the measure $Q \in \mathcal{M}_V^e$ relative to the basic measure P.

In the paper [1]a new approach was developed, where a characterization of optimal strategies in terms of a FBPDE system in the Brownian framework was given. The key observation was an existence of a stochastic process Y with $Y_T = H$ such that $U'(X_t^* + Y_t)$ is a martingale. The same approach was used in [3], where similar results were obtained in semimartingale setting with continuous filtration, rejecting some technical and not natural conditions imposed in [1]. We derive an other version of the BSPDE system, where the backward component P_t is a process, such that $P_t + U'(X_t^*)$ is a martingale.

It follows from [2] that under assumptions (1) and (2) optimal solutions $\pi^* \in \Pi$ and $Q^* \in \mathcal{M}_V^e$ to (3) and (4) exist and are related as

$$U'\left(x + \int_0^T \pi_s^* dS_s + H\right) = yZ_T^*, \quad P - a.s, \tag{5}$$

for a constant $y > 0$, where $Z_t^* = dQ_t^*/dP_t$ is the density process.

Theorem 1. *Let the utility function U be three-times continuously differentiable and let S a continuous semimartingale. Assume that conditions (1) and (2) are satisfied and $EU'(X_T^* + H) < \infty$. Then there exists a quadruple (P, ψ, L, X) that satisfies the FBSDE*

$$X_t = x - \int_0^t \frac{\lambda_s P_s + \lambda_s U'(X_s) + \psi_s}{U''(X_s)} dS_s, \tag{6}$$

$$\begin{aligned} P_t = P_0 + \int_0^t \Big[&\lambda_s \big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big) \\ &- \frac{1}{2} U'''(X_s) \frac{\big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big)^2}{U''(X_s)^2} \Big] d\langle M \rangle_s \\ &+ \int_0^t \psi_s dM_s + L_t, \quad P_T = U'(X_T^* + H) - U'(X_T^*). \end{aligned} \tag{7}$$

In addition the optimal strategy is expressed as

$$\pi_t^* = -\frac{\lambda_t P_t + \lambda_t U'(X_t^*) + \psi_t}{U''(X_t^*)} \tag{8}$$

and the optimal wealth X^* coincides with X.

Proof. Define the process

$$P_t = E(U'(X_T^* + H)/F_t) - U'(X_t^*). \tag{9}$$

It is evident that $P_T = U'(X_T^* + H) - U'(X_T^*)$.

Since U is three-times differentiable, $U'(X_t^*)$ is a continuous semimartingale and P_t admits the decomposition

$$P_t = P_0 + A_t + \int_0^t \psi_u dM_u + L_t, \tag{10}$$

where A is a predictable process of finite variations and L is a local martingale orthogonal to M.

Since Z_t^* is the density of a martingale measure , it is of the form $Z_t = \mathcal{E}_t(-\lambda \cdot M + R), R \perp M$. Therefore, (5) and (9) imply that

$$\begin{aligned} E(U'(X_T^* + H)/F_t) = yZ_t^* &= y - \int_0^t \lambda_s y Z_s^* dM_s + \tilde{R}_t \\ &= y - \int_0^t \big(P_s + U'(X_s^*)\big)\lambda_s dM_s + \tilde{R}_t, \end{aligned} \tag{11}$$

where $y = EU'(X_T^* + H)$ and $\tilde{R}$ is a local martingale orthogonal to M.

By definition of the process Y, using the Itô formula for $U'(X_t^*)$ and taking decompositions (10), (11) in to mind, we obtain

$$\begin{aligned} P_0 + A_t + \int_0^t \psi_s dM_s + L_t + y + \int_0^t \big(P_s + U'(X_s^*)\big)\lambda_s dM_s + \tilde{R}_t \\ = U'(x) + \int_0^t U''(X_s^*)\lambda_t \pi_s^* d\langle M\rangle_s + \frac{1}{2}\int_0^t U'''(X_s^*)\pi_s^{*2} d\langle M\rangle_s \\ + \int_0^t U''(X_s^*)\pi_s^* dM_s. \end{aligned} \tag{12}$$

Equalizing the integrands of stochastic integrals with respect to dM we have that $\mu^{\langle M\rangle}$-a.e.

$$\pi_t^* = -\frac{\lambda_t P_t + \lambda_t U'(X_t^*) + \psi_t}{U''(X_t^*)} \tag{13}$$

Equalizing the parts of finite variations in (12) we get

$$A_t = -\int_0^t \Big(U''(X_s^*)\lambda_s \pi_s^* + \frac{1}{2}U'''(X_s^*)\pi_s^{*2}\Big) d\langle M\rangle_s \tag{14}$$

and from (13), substituting the expression for π^* in (14) we obtain that

$$A_t = \int_0^t \big[\lambda_s\big(\lambda_s P_s + \lambda_s U'(X_s^*)\big)\big) - \frac{1}{2}U'''(X_s^*)\frac{\big(\lambda_s P_s + \lambda_s U'(X_s^*) + \psi_s\big)^2}{U''(X_s^*)^2}\big] d\langle M\rangle_s \tag{15}$$

Therefore, (15) and (10) imply that Y satisfies equation (7). Integrating both parts of equality (13) with respect to dS and adding the initial capital we obtain equation (6) for the optimal wealth. □

Corollary 1. *Let conditions of Theorem 1 be satisfied and assume that the filtration F is continuous. If the pair (X, P) is a solution of (6),(7), then the pair (X, Y), where*

$$Y_t = -V'(P_t + U'(X_t)) - X_t,$$

satisfies the FBSDE of [1] (see the martigale version in [3])

$$X_t = x - \int_0^t \frac{\lambda_s U'(X_s + Y_s) + Z_s U''(X_s + Y_s)}{U''(X_s + Y_s)} dS_s, \tag{16}$$

$$Y_t = Y_0 + \int_0^t \left(|\lambda_s|^2 \frac{U'(X_s + Y_s)}{U''(X_s + Y_s)} - \frac{1}{2}|\lambda_s|^2 \frac{U'''(X_s + Y_s)|U'(X_s + Y_s)|^2}{U''(X_s + Y_s)^3} + Z_s\lambda_s\right) d\langle M\rangle_s$$
$$-\frac{1}{2}\int_0^t \frac{U'''(X_s + Y_s)}{U''(X_s + Y_s)} d\langle \tilde{L}\rangle_s + \int_0^t Z_s dM_s + \tilde{L}_t, \quad Y_T = H. \tag{17}$$

Conversely, if the pair (X, Y) solves the FBSDE (16),(17), then $(X_t, P_t = U'(X_t + Y_t) - U'(X_t))$ satisfies (6),(7).

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