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STOCHASTIC DIFFERENTIAL EQUATIONS IN A BANACH SPACE, MAIN DIRECTIONS AND NEW METHOD OF DEVELOPMENT

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Abstract. The stochastic differential equation in a separable Banach space is considered. The main problem in traditional approach, the main directions and new method of development of the stochastic differential equations are investigated.

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1 Introduction. First results on the infinite dimensional stochastic differential equations started to appear in the mid 1960s. The traditional finite dimensional methods gave desired results for Hilbert space case (see [3], [4]), but they tuned out deadlock in the general Banach space case. Then, researchers began to develop the problem in such Banach spaces, the geometry of which is close to the geometry of the Hilbert space (see for example [1], [5]). Important results are received in the case, when the Banach space has UMD property (see [2], [6], [13]). But the class of UMD Banach spaces is very narrow—they are reflexive Banach spaces. Stochastic analysis in UMD Banach spaces intensively developed after the end of the eighties of the last century, but the class of Banach spaces, where the traditional methods give desired results, has not yet extended.

The first main problem to investigate stochastic differential equations in infinite dimensional spaces is to construct the Ito stochastic integral. The Ito stochastic integral in 2-uniformly smooth Banach spaces is considered in [2]. Stochastic integral in UMD Banach spaces is constructed in [6], [12].

We define the generalized stochastic integral in an arbitrary separable Banach space for a wide class of predictable random processes, which is a generalized random element (a random linear function or a cylindrical random element), if there exists the corresponding Banach space valued random element, that is, if this generalized random element is represented by the Banach space valued random element, then we say that this random element is the stochastic integral. Thus, the problem of existence of the stochastic integral in an arbitrary separable Banach space is reduced to the well known problem of decomposability (induction) of the generalized random element.

The second main problem to develop the stochastic differential equations in infinite dimensional space is to estimate the stochastic integral, which is necessary for the iteration procedure to prove the existence of the solution. Such estimations do not exist yet in a general Banach space case. We consider the Banach space of generalized random elements and introduce there the stochastic differential equation for the generalized random process.

For this situation, it is possible to use traditional methods to develop the problem of existence and uniqueness of the solution as a generalized random process. Afterward, from the main stochastic differential equation in an arbitrary Banach space we produce the equation for a generalized random process. As we have proved the existence and uniqueness of the solution of this equation, we have the generalized random process as a solution of the produced stochastic differential equation. If this generalized random process is decomposable, then the corresponding Banach space valued random process will be the solution of the main stochastic differential equation in a Banach space.

The investigation of the stochastic differential equations in a Banach space takes place in three directions. They can be described by means of the corresponding stochastic integrals in the equation. In the first (relatively) direction, the integrand predictable process takes its values in a Banach space and the stochastic integral is taken by the scalar Wiener process. We considered this case in [8], [9]. In the second direction the integrand predictable process is operator-valued (from Banach space to Banach space) and the stochastic integral is taken by the Wiener process in a Banach space. This case we investigated in the articles [7]. In the third direction the integrand is an operator-valued predictable random function from the Hilbert space to the Banach space while the stochastic integral is taken by the cylindrical Wiener process in a Hilbert space. We considered this case in [10]. For the evidence of demonstration of our method, in this article we consider the simplest– first case, as our method is same for all cases. Now we give some definitions to realize our approach.

2 Content.

Let X be a real separable Banach space, X^* is its conjugate, $B(X)$ is the Borel σ-algebra of X. (Ω, B, P) is a probability space. Continuous linear operator $L : X^* \to L_2(\Omega, B, P)$ is called a generalized random element (GRE). We consider such GRE, which maps X^* to a fix closed separable subspace $G \subset L_2(\Omega, B, P)$. Denote $M_1 := L(X^*, G)$ — the Banach space of GRE with the norm $\|L\| = \sup_{\|x^*\| \le 1} \|Lx^*\|_{L_2}$. A random element (measurable map) $\xi : \Omega \to X$ is said to have a weak second order if for all $x^* \in X^*$, $E\langle \xi, x^* \rangle^2 < \infty$. The random element ξ we can realize as an element of M_1, $L_\xi x^* = \langle \xi, x^* \rangle$ Denote by M_2 the linear space of all random elements of weak second order with the norm $\|\xi\| = \|L_\xi\|$. Therefore, we have $M_2 \subseteq M_1$.

Let $(W_t)_{t\in[0,1]}$ be a one-dimensional Wiener process and let $(F_t)_{t\in[0,1]}$ be such, that $(W_t)_{t\in[0,1]}$ is adapted to $(F_t)_{t\in[0,1]}$. Let $(L_t)_{t\in[0,1]}$ be a family of GRE. We call it a generalized random process (GRP).

Let $(L_t)_{t\in[0,1]}$ be adapted to the family of σ-algebra $(F_t)_{t\in[0,1]}$. Let TM_1 be the Banach space of adapted GRP $(L_t)_{t\in[0,1]}$, for which $\|L_t\| := \sup_{\|x^*\| \le 1} \left(\int_0^1 E(L_t x^*)^2 \, dt \right)^{1/2} < \infty$.

For any $x^* \in X^*$ we can define the scalar stochastic integral $\int_0^1 L_t x^* \, dW_t$.

Definition 1. For any adapted $(L_t)_{t\in[0,1]} \in TM_1$ the operator $I(L) : X^* \to L_2(\Omega, B, P)$, $I(L)x^* = \int_0^1 L_t x^* \, dW_t$ is called the generalized stochastic integral (GSI) of $(L_t)_{t\in[0,1]}$.

Definition 2. The stochastic integral from GRP $(L_t)_{t\in[0,1]}$ is called the X-valued random element $\eta : \Omega \to X$ (if such an element exists), for which $\langle \eta, x^* \rangle = I(L)x^*$ a.s. for all $x^* \in X^*$, $\eta := \int_0^1 L_t \, dW_t$.

Theorem 1. *Let the adapted GRP* $L : [0,1] \to M_1$ *be such, that* $L(t) \in \overline{M_2}$, *let the set* $(L(t), t \in [0,1])$ *be separable in* $\overline{M_2}$ *and let* $\int_0^1 \|L(t)\|_{M_1}^2 \, dt < \infty$. *Then there exists the sequence of adapted step functions* $f_n : [0,1] \times \Omega \to X$ *such, that* $\int_0^1 \|f_n(t) - L(t)\|_{M_1}^2 \, dt \to 0$, $\| \int_0^1 f_n(t) dW(t) - \int_0^1 L(t) dW(t)\|_{M_1}^2 \to 0$, *and, therefore,* $\int_0^1 L(t) \, dW(t) \in \overline{M_2}$.

Consider the stochastic differential equations for GRP.

$dT_t = a(t, T_t) \, dt + b(t, T_t) \, dW_t$ (1)

with F_0-measurable initial value $T_0 = T$ (Tx^* is F_0-measurable for all $x^* \in X^*$), where $a : [0,1] \times M_1 \to M_1$, $b : [0,1] \times M_1 \to M_1$, for all $x^* \in X^*$ the functions $a(t, T_t)x^*$ and $b(t, T_t)x^*$ are adapted and $E \int_0^1 |a(t, T_t)x^*|^2 dt + E \int_0^1 |b(t, T_t)x^*|^2 dt < \infty$.

Let us state a theorem on existence and uniqueness of a strong generalized solution to a stochastic differential equation for the GRP.

Theorem 2. *Let the functions* a *and* b *satisfy the following conditions:*

1) $\|a(t,T)\|_{M_1}^2 + \|b(t,T)\|_{M_1}^2 \le K^2(1 + \|T\|_{M_1}^2)$,

2) $\|a(t,T) - a(t,L)\|_{M_1}^2 + \|b(t,T) - b(t,L)\|_{M_1}^2 \le K^2 \|T - L\|_{M_1}^2$.

Then there exists a unique strong generalized solution $(T_t)_{t \in [0,1]}$ *to (1)*

$$T_t x^* = T x^* + \int_0^t a(s, T_s)x^* \, ds + \int b(s, T_s) x^* \, dW_s$$

a. s. for all $x^* \in X^*$, *with the initial condition* $T_0 = T$. *At the same time* $T_\cdot : [0,1] \to M_1$ *is continuous.*

Let us now consider the stochastic differential equation in a separable Banach space X:

$$d\xi_t = a(t, \xi_t) \, dt + b(t, \xi_t) dW_t, (2)$$

where $a : [0,1] \times X \to X$, $b : [0,1] \times X \to X$ are $B[0,1] \times B(X)$-measurable functions such, that

1') $\|a(t,\xi)\|_{M_1}^2 + \|b(t,\xi)\|_{M_1}^2 \le K^2(1 + \|\xi\|_{M_1})^2$,

2') $\|a(t,\xi) - a(t,\eta)\|_{M_1}^2 + \|b(t,\xi) - b(t,\eta)\|_{M_1}^2 \le K^2 \|\xi - \eta\|_{M_1}^2$,

where ξ, η are random elements with values in X.

If conditions 1') and 2') are satisfied, then we can extend coefficients a and b to $\overline{M}_2 \subseteq M_1$.

Therefore, from equation (2) we receive the stochastic differential equation for GRP

$$dT_t = a(t, T_t) \, dt + b(t, T_t) \, dW_t$$

with F_0-measurable initial condition $T_0 x^* = \langle \xi_0, x^* \rangle$, where the measurable adapted functions a and b satisfy the conditions $1'$) and $2'$). Therefore, the following theorem is true.

Theorem 3. *Let the functions a and b satisfy conditions 1'), 2') and for all random element $\xi \in M_2$ the maps $a(\cdot,\xi)$ and $b(\cdot,\xi)$ from $[0,1]$ to M_1 are continuous. Then the stochastic differential equation (2) has a unique generalized solution T_t with the initial condition $T_0 x^* = \langle \xi_0, x^* \rangle$, where $\xi_0 \in M_2$ is F_0-measurable random element. The solution T_t is such, that $T_t \in \overline{M_2}$.*

In [9] we give the generalized solutions of the linear stochastic differential equations. If these generalized solutions satisfy conditions of decomposability (see [8],Theorems 2-4), then they will be the solutions of the linear stochastic differential equations in a Banach space.

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