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SUPERSYMMETRIC DYNAMICS AND QUANPUTERS

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Abstract. Functional formulation of quantum, classical and stochastic dynamics and supersymmetry; formal, long wave and short wave solutions of thecanonical equation for generalized analitic functions and theory of quanputers considered.

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1 Quantum, classical and stochastic dynamics and supersymmetry. After formulation of the mathematical framework of quantum mechanics (QM), operatorial formulation of QM, Koopman and von Neumann gave operatorial approach to classical Hamiltonian mechanics [5], [11]. After Wiener introduction of the functional integrals, Dirac and Feynman gave formal functional integral formulation of the quantum theory [2]. Gozzi invented functional integral formulation of the classical theory [4]. All stochastic and deterministic differential equations, describing all natural and engineered dynamical systems, possess a topological supersymmetry. Its spontaneous breakdown could be interpreted as the stochastic generalization of deterministic chaos. This conclusion stems from the fact that such phenomenon encompasses features that are traditionally associated with chaotic dynamics such as non-integrability, positive topological entropy, sensitivity to initial conditions. Spontaneous topological symmetry breaking can be considered as the most general definition of continuous-time dynamical chaos. For supersymmetric gauge theories stochastic quantization appears to have one definite advantage: since a gauge fixing term is unnecessary, supersymmetry will not be broken at any step. This holds both for the Abelian and non-Abelian case. It appears at the moment as if stochastic regularization is the only viable candidate for a regularization scheme which manifestly conserves both supersymmetry, chiral symmetry and gauge invariance. However, supersymmetry is related to stochastic quantization also at a much deeper level. As an example, even purely scalar field theories will, when quantized stochastically, display a 'hidden' supersymmetry. This issue, is intimately connected with the existence of a so-called 'Nicolai map' for supersymmetric field theories [8]. Parisi-Sourlas 'dimensional reduction' of scalar field theories in external random fields [9], is closely related to both supersymmetry and stochastic quantization. This becomes apparent when one establishes the connection to the Nicolai map. The phenomenon of dynamical 'dimensional reduction' was first noted within the context of critical phenomena associated with spin systems in random external fields. Systems very close to such a situation can in fact be created and studied in the laboratory. Let us concider the Langevin equation associated with a point particle being subjected to random background noise. This corresponds to the very real physical problem of the Brownian motion of a (classical) particle in a heat bath. The Langevin equation for the particle reads

$$\frac{dx}{dt} \equiv \dot{x} = -\frac{\delta S}{\delta x} + \eta(t), \tag{1}$$

where x represents the space coordinate of the particle. Expectation values are, as usual, evaluated as the path integral

$$< x(t_1)... \; x(t_n) >= \int d\eta \; x(t_1)... \; x(t_n) \exp(-\frac{1}{4}\int dt\eta(t)^2) \tag{2}$$

over a Gaussian noise, i.e.

$$< \eta(t_1)\eta(t_2) >= 2\delta(t_1 - t_2). \tag{3}$$

We now attempt to make a change of variables: $\eta \to x$. This involves the Jacobian

$$det(\delta\eta(t)/\delta x(t')) = det((d/dt + V')\delta(t - t')), \; V = \delta S/\delta x. \tag{4}$$

For partition function Z, we have

$$\begin{aligned} Z &= \int d\eta \exp(-\frac{1}{4}\int dt\eta(t)^2) \\ &= \int d\eta dx det(d/dt + V')\delta(\dot{x} + V - \eta(t)) \exp(-\frac{1}{4}\int dt\eta(t)^2) \\ &= \int dx det(d/dt + V') \exp(-\frac{1}{4}\int dt(\dot{x} + V)^2) \\ &= \int dx d\psi d\bar{\psi} \exp(-S), \; S = \int dt(\frac{1}{4}(\dot{x} + V)^2 - \bar{\psi}(d/dt + V')\psi). \end{aligned} \tag{5}$$

This system is recognized as Witten's example of supersymmetric quantum mechanics. So, purely classical (although stochastic) problem of Brownian motion is completely equivalent to a (supersymmetric) quantum mechanical problem!

2 Generalized or pseudoanalytic functions (GPF). The theory of analytic functions of a complex variable occupies a central place in analysis. Riemann considered the unique continuation property to be the most characteristic feature of analytic functions. GPF do possess the unique continuation property, and each class of GPF has almost as much structure as the class of analytic functions. In particular, the operations of complex differentiation and complex integration have meaningful counterparts in the theory of GPF and this theory generalizes not only the Cauchy-Riemann approach to function theory but also that of Weierstrass. Such functions were considered by Picard and by Beltrami, but the first significant result was obtained by Carleman in 1933, and a systematic theory was formulated by Lipman Bers [1] and Ilia Vekua (1907-1977), [10]. For more resent results see [3].

3 Formal, long-wave and short-wave solutions of the canonical equations for GPF. Analytic function $f = u + iv$ satisfy the partial differential equation $\partial_{\bar{z}} f = 0$, where complex differential operators are defined as

$$\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\partial_x + i\partial_y),\ \partial_z = \frac{\partial}{\partial z} := \frac{1}{2}(\partial_x - i\partial_y) \tag{6}$$

Generalized analytic functions $f = u + iv$ satisfy the following generalized Cauchy-Riemann equation [10]

$$\partial_{\bar{z}} f = Af + B\bar{f} + J,\ A = A_0 + iA_1,\ B = B_0 + iB_1,\ J = j_1 + ij_2 \tag{7}$$

or in terms of the real u and imaginary v components canonical form of the elliptic systems of partial differential equations of the first order

$$\begin{aligned} u_x - v_y &= au + bv + j_1,\ a = A_0 + B_0,\ b = -A_1 + B_1, \\ u_y + v_x &= cu + dv + j_2,\ c = A_1 + B_1,\ d = A_0 - B_0, \end{aligned} \tag{8}$$

or in the matrix form

$$\begin{aligned} & D\psi = E\psi + J,\ D = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix} = \partial_x - i\sigma_2\partial_y, \\ & E = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \psi = \begin{pmatrix} u \\ v \end{pmatrix},\ J = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}. \end{aligned} \tag{9}$$

In the classical sense by a solution of the system of equations (9) we understand a pair of real continuously differentiable functions $u(x, y),\ v(x, y)$ of the real variables x and y which satisfy this system everywhere in a domain G. Such solutions, however, exist only for a comparatively narrow class of equations.

The formal solution of the canonical equation for GPF (9) is

$$\psi = \psi_0 + RJ,\ R = (D - E)^{-1},\ (D - E)\psi_0 = 0. \tag{10}$$

Let us introduce a 'fundamental' length parameter $l = h^{-1}$, $\mathbf{x}_n = lx_n,\ n = 1, 2,\ x_1 = x,\ x_2 = y,\ x_n$ is dimensionless. Then, for the resolvent R, we will have the long-wave and short-wave expansions,

$$\begin{aligned} & R_{LW} := (lD - E)^{-1} = -E^{-1} \sum_{n\geq 0} l^n (DE^{-1})^n, \\ & R_{ShW} := (lD - E)^{-1} = hD^{-1} \sum_{n\geq 0} h^n (ED^{-1})^n, \\ & E^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} /\Delta_E,\ \Delta_E = ad - bc, \\ & D^{-1} = \Delta_D^{-1} \begin{pmatrix} \partial_x & \partial_y \\ -\partial_y & \partial_x \end{pmatrix},\ \Delta_D = \partial_x^2 + \partial_y^2 \end{aligned} \tag{11}$$

There is a fairly complete theory of generalized analytic functions; it represents an essential extension of the classical theory preserving at the same time its principal features [10].

4 Hamiltonization of dynamical systems and quanputers. The following system of the ordinary differential equations

$$\dot{x}_n = v_n(x) + j_n(t),\ 1 \leq n \leq N, \tag{12}$$

Using the following Lagrangian,

$$L = (\dot{x}_n - v_n(x) - j_n(t))\psi_n \tag{13}$$

and the corresponding motion equations

$$\dot{x}_n = v_n(x) + j_n(t), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m, \tag{14}$$

we extended by linear equation for the variables ψ, which maybe complex as well as grassmann valued, with corresponding supersymmetry. The extended system can be put in the Hamiltonian form [6]. Infinite dimensional as well as discrete analogues of this system are basis of the quanputers [7].

REFERENCES

1. Bers, L. Theory of Pseudo-Analytic Functions. *New York University*, 1952.
2. Feynman, R.P., Hibbs R.A. Quantum Mechanics and Path-Integrals. *McGraw-Hill, Inc.*, 1965.
3. Edited by Giorgadze G. Recent Developments in Generalized Analytic Functions and Their Applications. *Proceedings of the International Conference on Generalized Analytic Functions and Their Applications,* TSU, Tbilisi, 2011.
4. Gozzi, E. Functional techniques in classical mechanics. *Nucl. Phys. Proc. Suppl.,* **104** (2001), 243-246.
5. Koopman, B.O. Hamiltonian systems and transformations in Hilbert space. *Proc. Nat. Acad. Sci. USA,* **17**, 5 (1931), 315-318.
6. Makhaldiani, N. Nambu-Poisson dynamics with some applications. *Physics of Particles and Nuclei,* **43** (2012), 1357-1367.
7. Makhaldiani, N. Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers. *Atomic Nuclei,* **74**, (2011), 1040.
8. Nicolai, H. On a New Characterization of Scalar Supersymmetric Theories. *Phys. Lett. B,* **89** (1980), 341-346.
9. Parisi, G., Sourlas N. Random magnetic fields, supersymmetry, and negative dimensions. *Phys. Rev. Lett.,* **43**, 11 (1979), 744-745.
10. Vekua, I.N. Generalized Analytic Functions (Russian). *Moscow: Nauka*, 1959; English translation: *Oxford: Pergamon Press,* 1962.
11. von Neumann, J. Zur operatorenmethode in der klassichen mechanik. *Ann. Math.,* **33**, 3 (1932), 587-642.

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