Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

SYMBOLIC EXPRESSION FOR LOCATION REGION OF REAL (IMAGINARY) PARTS OF AN ONE-VARIABLE POLYNOMIAL'S ROOTS AND ESTIMATION OF THE MINIMAL DISTANCE BETWEEN THEM

Ilia Lomidze Natela Chachava

Abstract. The method for accurate symbolic estimation of maximal and minimal meanings of real (imaginary) parts of roots of any one-variable polynomial is found out. Only rational functions of the coefficients of the polynomial are used. The same method allows to estimate the minimal distance between real (imaginary) parts of the roots.

Keywords and phrases: Polynomials, roots location, Gershgorin disks.

AMS subject classification (2010): 12Y99, 47B15.

1 Introduction. We have proposed in [1], [2] a method for symbolic description of a polynomial's roots location, if all roots are real. In the present article we generalize this result for any polynomial. Problem of polynomial's roots location has long history (see, e.g. [3] and sources citied there). Up to our knowledge, most of used methods describe the region of polynomial's roots as unity of some circles on complex plane. These (quite rough) methods need numerical fitting to get necessary accuracy. In contrast, we symbolically describe the region of the real parts and, separately, of the imaginary parts location with any accuracy. Besides, we estimate a minimal distance between real (imaginary) parts of different roots.

2 Theorem. *For any polynomial* $P_n(z) = z^n + \sum_{k=1}^{n} (-1)^k c_k z^{n-k} \in \mathbb{C}[z]$ *having roots*

$$z_k \in \{r_q;\; a_l + ib_l | q = \overline{1, n_1}, l = \overline{1, n_2}, n_1 + n_2 = n; r_q,\, a_l, b_l \in \mathbb{R}; b_l \neq 0\}, \tag{1}$$

a) bounded above sequence

$$\mu_0 < \mu_1 < ... < \mu_l < \min\{r_j, a_l | j = \overline{1, n_1}, l = \overline{1, n_2}\} \tag{2}$$

(bounded bellow sequence $\eta_0 > \eta_1 > ... > \eta_l > \max\{r_j, a_l | j = \overline{1, n_1}, l = \overline{1, n_2}\}$*) can be constructed using only rational functions of* $P_n(z)'s$ *coefficients. b)* $\lim_{p\to\infty} \mu_p = \min\{r_j, a_l | j = \overline{1, n_1}, l = \overline{1, n_2}\}$ $(\lim_{p\to\infty} \eta_p = max\{r_j, a_l | j = \overline{1, n_1}, l = \overline{1, n_2}\})$.

c) The similar statements are correct for imaginary parts of the roots.

Proof. a) Obviously,

$$(z_k + z_j^*)/2 \in \{R_q, A_l \pm iB_l | q = \overline{1, m_1}, l = \overline{1, m_2}, m_1 + 2m_2 = n^2\}, \tag{3}$$

where $k, j = \overline{1, n},\ z^*$ denotes complex conjugate of z and

$$\begin{aligned} R_q &\in \{r_j, a_l, (r_j + r_i)/2, (a_l + a_m)/2\ (if\, b_l = b_m)\}, \\ A_l \pm iB_l &\in \{(r_j + a_m)/2 \pm ib_m/2, (a_l + a_m)/2 \pm i(b_l - b_m)/2\}; \\ &(j, i = \overline{1, n_1}, l, m = \overline{1, n_2}, n_1 + n_2 = n) \end{aligned} \quad (4)$$

$$\min\{R_k\} \stackrel{\text{def}}{=} \underline{R} = \min\{r_q, a_l\} \leq \min\{A_l\};\ \max\{R_k\} \stackrel{\text{def}}{=} \overline{R} = \max\{r_q, a_l\} \geq \max\{A_l\}. \quad (5)$$

Let us construct the following sum

$$M(\mu) \stackrel{\text{def}}{=} \sum_{k,\, l} \frac{1}{(z_k + z_l^*)/2 - \mu} = \sum \frac{1}{R_k - \mu} + \sum \frac{2(A_l - \mu)}{(A_l - \mu)^2 + B_l^2} \quad (6)$$

that can be expressed by rational functions of P_n's coefficients only. Obviously, if $\mu < \underline{R}$ then $M(\mu) > 0$ and, besides, all summands in (6) are positive. Therefore

$$M(\mu) > (\underline{R} - \mu)^{-1} \quad \Rightarrow \quad \underline{R} > \mu + M^{-1}(\mu). \quad (7)$$

Using the well-known theorem for polynomial's roots location [4] one gets for $P_n(z)$:

$$|z_k| \leq 1 + \max\{|c_i|\}. \quad (8)$$

Supposing in (6) and (7) $\mu = \mu_0 < -(1 + \max\{|c_i|\}) \leq \underline{R}$ one can construct the sequence (2) using a recurrence relation

$$\mu_{p+1} \stackrel{\text{def}}{=} \mu_p + M^{-1}(\mu_p) < \underline{R} = \min\{r_q, a_l\}. \quad (9)$$

b) Calculating the limit in (9) one gets:

$$\mu_\infty \stackrel{\text{def}}{=} M^{-1}(\mu_\infty) + \mu_\infty \Rightarrow M^{-1}(\mu_\infty) = \left(\sum \frac{1}{R_k - \mu_\infty} + \sum \frac{2(A_l - \mu_\infty)}{(A_l - \mu_\infty)^2 + B_l^2}\right)^{-1} = 0. \quad (10)$$

Hence, in the last sum at least one summand is infinite - at least one denominator in it equals to 0. Taking into account that $0 < \mu_p < \underline{R}$ for any p, one has to conclude $\mu_\infty = \underline{R}$.

To construct the sequence $\eta_0 > \eta_1 > ... > \eta_l > \overline{R}$, one can consider the following sum:

$$N(\eta) \stackrel{\text{def}}{=} \sum_{k,l} \frac{1}{(z_k + z_l^*)/2 - \eta} = \sum \frac{1}{R_k - \eta} + \sum \frac{2(A_l - \eta)}{(A_l - \eta)^2 + B_l^2}. \quad (11)$$

Choosing

$$\eta = \eta_0 > 1 + \max\{|c_i|\} \geq \overline{R}, \quad i = \overline{1, n},$$

one concludes that $N(\eta_0) < 0$ and, besides, that all summands in (11) are negative. Hence, analogically to (9) and (10) one gets:

$$\overline{R} < \eta_{p+1} \stackrel{\text{def}}{=} \eta_p + N^{-1}(\eta_p) < \eta_p, \quad \eta_\infty \stackrel{\text{def}}{=} \lim_{p \to \infty} \eta_p = \overline{R}. \quad (12)$$

c) The similar sequences can be constructed for imaginary parts of the roots. It is enough to consider the polynomial $P'_n(z) \stackrel{\text{def}}{=} i^{-n} P_n(iz)$. □

3 To estimate the minimal distance between the real parts of roots (1) one has to construct some new auxiliary polynomials: the polynomial

$$Q_{n^2}(u) \overset{\text{def}}{=} 2^{-n^2} \prod_{k=1}^{n} P_n(2u - z_k^*) \tag{13}$$

that has roots (4), and the polynomial

$$K_{n^4}(v) \overset{\text{def}}{=} \prod_{k=1}^{n^2} Q_{n^2}(v + u_k) = v^{n^4} + \sum_{k=1}^{n^4} (-1)^k d_k v^{n^4-k} \tag{14}$$

having roots:

$$\begin{gathered} v_k \in \{0,\ \pm(r_j - r_i),\ \pm(a_l - a_m),\ \pm(r_j - a_m), \\ \pm \textit{arithm. m.};\ \pm(\textit{complex roots})\}. \end{gathered} \tag{15}$$

Here arithmetical means of pairs from the set $\{(r_j - r_i),\ (a_l - a_m), (r_j - a_m)\}$ are meant.

Supposing that one has for the coefficients in (14)

$$d_{n^4} = 0, d_{n^4-1} = 0,\ ...,\ d_{n^4-k_1+1} = 0, d_{n^4-k_1} \neq 0, k_1 \geq n^2,$$

one can determine the number of zero roots k_1 of $K_{n^4}(v)$. Hence, the new polynomial

$$V_{n^4-k_1}(v) \overset{\text{def}}{=} v^{-k_1} K_{n^4}(v) \tag{16}$$

has no zero roots. Therefore one can construct the polynomial

$$S_{n^4-k_1}(t) \overset{\text{def}}{=} (-1)^{n^4-k_1} t^{n^4-k_1} d^{-1}_{n^4-k_1} V_{n^4-k_1}(t^{-1}) \tag{17}$$

having roots

$$t_k \in \{\pm(r_l - r_i)^{-1},\ \pm(a_l - a_m)^{-1},\ \pm(r_l - a_m)^{-1}, ...;\ \pm(\textit{complex roots})\}. \tag{18}$$

According to the Theorem, the sequence (2) for the last polynomial can be built. Hence,

$$\begin{gathered} \mu'_p < \min\{\pm(r_j - r_i)^{-1},\ \pm(a_l - a_m)^{-1},\ \pm(r_j - a_m)^{-1},\ ...\} < 0 \\ \Rightarrow |\mu'_p| > \max\{|r_j - r_i|^{-1},\ |a_l - a_m|^{-1}, |r_j - a_m|^{-1}, ...\} \end{gathered} \tag{19}$$

and

$$\min\{|r_j - r_i|, |a_l - a_m|, |r_j - a_m|, ...\} > |\mu'_p|^{-1}. \tag{20}$$

4 The same quantities μ_{p1}, η_{l1} and μ'_{p1} can be constructed for imaginary parts of the roots in a similar way. It is enough to construct the polynomial $P'_n(z) \overset{\text{def}}{=} i^{-n} P_n(iz)$.

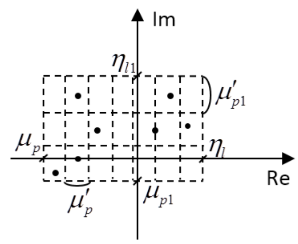

Figure 1: The grid

5 Conclusions. The grid including all roots of $P_n(z)$ can be created on the complex plane (Fig.1) using only rational functions of $P_n(z)$'s coefficients. The boarders $(\mu_p, \eta_l, \mu_{p1}, \eta_{l1})$ can be found symbolically with any accuracy whereas up to now quite rough estimation was given (see, e.g., [4-7] and (8) in the present article); these boundaries are described more precisely by Gershgorin disks [7], but only for eigenvalues of the matrix (that is, for its characteristic polynomial's roots), using full information on the matrix and then numerical methods are used to fit boundaries even for single root location. In each cell of our grid there is at most one root; for root that lies just on the boundary of the cell there are no roots in neighbouringr cells. In some cases our estimation of the minimal distance is not the best: the matter is that we calculate the minimal distance not only between real (imaginary) parts of roots but between all real members of the set (15).

R E F E R E N C E S

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Received 10.05.2017; revised 11.09.2017; accepted 13.10.2017.

Author(s) address(es):

Ilia Lomidze
Georgian Technical University
M. Koctava str. 77, 0175 Tbilisi, Georgia
E-mail: lomiltsu@gmail.com

Natela Chachava
Georgian Technical University
M. Koctava str. 77, 0175 Tbilisi, Georgia
E-mail: Chachava.Natela@yahoo.com