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ON ONE REDUNDANT SYSTEM AND ITS NUMERICAL SOLUTION

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Abstract. A system with m main and n stand-by units is considered, where repair and replacement operations are performed. The boundary value problem that contains integral boundary conditions is stated for the system of the first order partial differential equations which is under study. Results and analysis of the numerical experiments for the case $m = 3$ and $n = 2$ are given.

Keywords and phrases: Redundant System, repairman and replacement problem, numerical approximation

AMS subject classification (2010): 60K25, 60K20, 90B25, 68M15, 45K05, 65N06.

Reliability and Performability analysis for redundant systems is an important problem (see, for example, [8] – [13]). This problem is studied within frameworks of the mathematic theory of reliability and queuing theory.

Several new methods have been developed for the last years, which are the interest for analysis of modern technical systems (see, for example, [3] – [7]). One of such systems is described below.

The redundant system contains m main and n redundant identical units. Failure rate of main units are $\alpha \geqslant 0$. Redundant units do not fail. Failed main unit is replaced by operative unit by a replaceman. If there is not operative unit in reserve, replacement process starts by the time of end of the repair process. There is one replaceman. Distribution function for replacement time is arbitrary - F. If the replaceman is busy, units that must be replaced join a waiting line. Units do not fail in the waiting line. Failed unit moves to repairman. There is one repairman. Unit that came first will be repaired first. If repairman is busy, failed units join a waiting line. Repair time distribution is exponential with parameter μ. Repaired unit becomes identical to new one and joins redundant units or replacement waiting line, if replacement of this unit is necessary. Thus, we have a closed type queuing system with two types of services – replacement and repair.

Consider random processes.

- $k(t)$ – number of missed units from main units group at the moment t.
- $l(t)$ – number of failed units at the moment t.
- $\theta(t)$ – elapsed time from the start of last replacement operation at the moment t.

Below $\lambda(u)$ means intensity of replacement rate.

$$\lambda(u) = \frac{f(u)}{1 - F(u)},$$

where $f(u) = F'(u)$.

Denote:

- $P_l(t) = P\{k(t) = 0, l(t) = l\}, \quad l = \overline{0, n}$.
- $R_k(t) = P\{k(t) = k, l(t) = n + k\}, \quad k = \overline{0, m}$.
- $q_{k,l}(t, u) = \lim\limits_{h\to 0} \left(\frac{1}{h} P\{k(t) = k, l(t) = l - 1, u < \theta(t) < u + h\}\right), k = \overline{1, m}; l = \overline{1, n + k}$.

There are 3 systems for $P_l(t)$, $R_k(t)$ and $q_{k,l}(t, u)$ functions. We consider the System (1) of the following partial differential equations, that are satisfied by $q_{k,l}(t, u)$ functions.

1. $\dfrac{\partial q_{1,l}(t,u)}{\partial t} + \dfrac{\partial q_{1,l}(t,u)}{\partial u} = -\left((m-1)\alpha + \mu + \lambda(u)\right) q_{1,l}(t,u) + \mu q_{1,l+1}(t,u),\ 1 < l \leqslant n;$
2. $\dfrac{\partial q_{1,n+1}(t,u)}{\partial t} + \dfrac{\partial q_{1,n+1}(t,u)}{\partial u} = -\left((m-1)\alpha + \mu + \lambda(u)\right) q_{1,n+1}(t,u);$
3. $\dfrac{\partial q_{k,1}(t,u)}{\partial t} + \dfrac{\partial q_{k,1}(t,u)}{\partial u} = -\left((m-i)\alpha + \lambda(u)\right) q_{k,1}(t,u) + \mu q_{k,2}(t,u), \quad 1 \leqslant k \leqslant m;$
4. $\dfrac{\partial q_{k,l}(t,u)}{\partial t} + \dfrac{\partial q_{k,l}(t,u)}{\partial u} = -\left((m-i)\alpha + \mu + \lambda(u)\right) q_{k,l}(t,u)$
   $+ \left(m - (i-1)\right) \alpha q_{k-1,l-1}(t,u) + \mu q_{k,l+1}(t,u), \quad 1 < k \leqslant m; \quad 1 < l < n + k;$
5. $\dfrac{\partial q_{k,n+k}(t,u)}{\partial t} + \dfrac{\partial q_{k,n+k}(t,u)}{\partial u} = -\left((m-i)\alpha + \mu + \lambda(u)\right) q_{k,n+k}(t,u)$
   $+ \left(m - (i-1)\right) \alpha q_{k-1,n+k-1}(t,u), \quad 1 < k \leqslant m;$

Boundary conditions of $q_{k,l}(t, u)$ functions are written:

1. $q_{1,1}(t,0) = \displaystyle\int_0^t q_{2,1}(t,u)\lambda(u)du;$
2. $q_{1,l}(t,0) = \displaystyle\int_0^t q_{2,l}(t,u)\lambda(u)du + \mu\alpha P_{l-2}(t), \quad 1 < l < n + k;$
3. $q_{1,l}(t,0) = \displaystyle\int_0^t q_{2,l}(t,u)\lambda(u)du + \mu\alpha P_{l-2}(t) + \mu R_1(t), \quad l = n + 1;$
4. $q_{k,l}(t,0) = \displaystyle\int_0^t q_{k+1,l}(t,u)\lambda(u)du, \quad 1 < k < m; \quad 1 \leqslant l < n + k;$
5. $q_{k,l}(t,0) = \displaystyle\int_0^t q_{k+1,l}(t,u)\lambda(u)du + \mu R_k(t), \quad 1 < k < 1; \quad l = n + k;$
6. $q_{m,l}(t,0) = 0, \quad 1 < l < n + m;$
7. $q_{m,n+m}(t,0) = \mu R_m(t);$

Algorithm. Using methods of constructing of discrete analogs for differential and integro-differential models (see, for example, [1] and [2]) following difference scheme is built for system (1).

1. $q_{1,l}{}_{i+1}^{j+1} = \left(\dfrac{h}{\tau} - hr\right) q_{1,l}{}_i^j + \left(1 - \dfrac{h}{\tau}\right) q_{1,l}{}_i^{j+1} + h\mu q_{1,l+1}{}_i^j + hf_{1,l}{}_i^j,$
   $1 < l \leqslant n, \quad r = ((m-1)\alpha + \mu + \lambda(u));$

2. $q_{1,l}{}_{i+1}^{j+1} = \left(\dfrac{h}{\tau} - hr\right) q_{1,l}{}_i^j + \left(1 - \dfrac{h}{\tau}\right) q_{1,l}{}_i^{j+1} + hf_{1,l}{}_i^j,$
   $l = n+1, \quad r = ((m-1)\alpha + \mu + \lambda(u));$

3. $q_{k,1}{}_{i+1}^{j+1} = \left(\dfrac{h}{\tau} - hr\right) q_{k,1}{}_i^j + \left(1 - \dfrac{h}{\tau}\right) q_{k,1}{}_i^{j+1} + h\mu q_{k,2}{}_i^j + hf_{k,1}{}_i^j,$
   $1 \leqslant k \leqslant m, \quad r = ((m-k)\alpha + \lambda(u));$

4. $q_{k,l}{}_{i+1}^{j+1} = \left(\dfrac{h}{\tau} - hr\right) q_{k,l}{}_i^j + \left(1 - \dfrac{h}{\tau}\right) q_{k,l}{}_i^{j+1} + h\left(m - (k-1)\right)\alpha q_{k-1,l-1}{}_i^j + h\mu q_{k,l+1}{}_i^j$
   $+ hf_{k,l}{}_i^j, \quad 1 < k \leqslant m, \quad 1 < l < n+k, \quad r = ((m-k)\alpha + \mu + \lambda(u));$

5. $q_{k,l}{}_{i+1}^{j+1} = \left(\dfrac{h}{\tau} - hr\right) q_{k,l}{}_i^j + \left(1 - \dfrac{h}{\tau}\right) q_{k,l}{}_i^{j+1} + h\left(m - (k-1)\right)\alpha q_{k-1,l-1}{}_i^j + hf_{k,l}{}_i^j,$
   $1 < k \leqslant m, \quad l = n+k, \quad r = ((m-k)\alpha + \mu + \lambda(u)).$

Boundary Conditions of $q_{k,l}(t,u)$ are rewritten as follows:

1. $q_{k,l}(i,0) = \sum_{c=0}^{i-1} q_{k+1,l}(i,c) + b_{k,l}(i,0)$, when $k < m$.
2. $q_{k,l}(i,0) = b_{k,l}(i,0)$, when $k = m$.

We have got a large system of equations. The boundary conditions allow to find values of functions q_k, l at points $(i;0)$ and the other equations allow to find values of the same functions at the remaining points.

Numerical Solution. At first, we do $k = m, l = n+k$, when $i = 0, j = 0$. After that l varies from $n+k$ to 1, for each k, when k varies from m to 1, for each j, when j varies from 0 to d, for each i, when i varies from 0 to d.

Thus, we get four independent nested loops.

Experiment.

$$q_{k,l} = \frac{\sin(35t + 22u)}{k+l}$$

Parameters: $m = 3; n = 2; \alpha = 1; \mu = 1; \lambda = 1; t = 1; u = 1; d = 3000.$

Result: Maximum difference between real and algorithm result numbers is 0.003320141.

$q_{3,5}(i',j')$	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(3,12)
Real	0.034962	0.106305	0.122995	0.076453	-0.009394	-0.059053
Algorithm	0.033842	0.105334	0.122387	0.0773	-0.007145	-0.058366

(8,12)	(9,12)	(10,12)	(11,12)	(10,20)	(11,20)	(12,20)
0.106324	0.122988	0.076424	-0.009431	-0.085137	-0.001991	0.082179
0.106016	0.124538	0.076588	-0.008872	-0.084944	-0.003203	0.080582

(20,24)	(21,24)	(2,28)	(3,28)	(4,28)	(5,28)	(22,30)
-0.120953	-0.068743	0.051447	-0.038031	-0.107957	-0.122382	0.095368
-0.120875	-0.067902	0.053144	-0.037771	-0.107749	-0.121828	0.094163

$$i' = i/100, j' = j/100.$$

We have done many other experiments. Results say that when $d \to \infty$, maximum difference value goes to 0.

REFERENCES

1. Jangveladze, T. Methods of Approximate Solution of Ordinary Differential Equations (Georgian). *University Press, Tbilisi*, 2005.
2. Jangveladze, T., Kiguradze, Z., Neta, B. Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations. *Academic Press, Elsevier*, 2016.
3. Kakubava, R. Multi-line markov closed queuing system for two maintenance operations. *RTA Journal*, **1** (2010), 15–22.
4. Kakubava, R. Reliability model for standby system with replacement delays of failed elements. *Automatic Control and Computer Sciences*, **47** (2013), 94–98.
5. Kakubava, R., Khurodze, R. Probabilistic analysis of the downtime of a duplicated system with recovery and switching. *Automation and Remote Control, Part 1*, **61** (2000).
6. Kakubava, R., Svanidze, N. The semi markovian model for economic analysis of standby. *Int. J. of Engineering, Science and Innovative Technology*, **2** (2013), 240-246.
7. Kakubava, R., Sztrik, J. Queuing models with two types of service: applications for dependability planning of complex system. *In Proc. VII International Conference. MMR 2011 - Mathematical Methods in Reliability. Theory. Methods. Applications, Beijing, China*, 2011.
8. Korolyuk, V. Stochastic Models of Systems. *Kluwer Academic Publisher, Dordrecht*, 1999.
9. Korolyuk, V., Limnios, N. Stochastic Systems in Merging Phase Space. *World Scientific, Imperial College Press, Singapore*, 2005.
10. Kovalenko, I., Pegg, P., Kuznetsov, N. Mathematical Theory of Reliability of Time Dependent Systems with Practical Applications. *John Wiley & Sons*, 1997.
11. Limnios, N., Oprisan, G. Semi-Markov Processes and Reliability. *Birkhauser, Boston*, 2001.
12. Ushakov, I. Reliability: past, present, future. *RTA Journal* (2006), 10-27.
13. Ushakov, I. Reliability theory: history & current state in bibliographies. *RTA Journal* (2012), 8-35.

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