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STRONG ACCRETIVE PROPERTY OF FRACTIONAL DIFFERENTIATION OPERATOR IN THE KIPRIYANOV SENSE

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Abstract. In this paper we will prove theorem establishes the strong accretive property for the operator of fractional differentiation in the Kipriyanov sense.

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1 Introduction.

The term accretive applicable to a linear operator T acting in the Hilbert space H was introduced by Friedrichs in [1], and means that the operator has the following property - a numeric domain of values $\Theta(T)$ is a subset of the right half-plane i.e.

$$\mathrm{Re}\langle Tu, u\rangle_H \geq 0,\ u \in \mathfrak{D}(T).$$

Accepting a notation [2] we assume that Ω is a convex domain of the n - dimensional Euclidean space, P is a fixed point of the boundary $\partial\Omega$, $Q(r, \vec{\mathbf{e}})$ is an arbitrary point of Ω; we denote by $\vec{\mathbf{e}}$ a unit vector having the direction from P to Q, using r is the Euclidean distance between points P and Q. We will consider classes of Lebesgue $L_p(\Omega),\ 1 \leq p < \infty$ complex valued functions. In polar coordinates summability f on Ω of degree p means that

$$\int\limits_{\Omega} |f(Q)|^p dQ = \int\limits_{\omega} d\chi \int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^p r^{n-1} dr < \infty, \tag{1}$$

where $d\chi$ is the element of the solid angle of the surface a unit sphere in the n-dimensional space and $\omega-$ is surface of this sphere, $d(\vec{\mathbf{e}})-$ is the length of segment of ray going from point P in the direction $\vec{\mathbf{e}}$ within the domain Ω. The set obtained by the intersection of the ray going from point P in the direction $\vec{\mathbf{e}}$, and domain Ω is denoted by $\Omega_{\vec{\mathbf{e}}}$. Without loses of generality, we consider only those directions of $\vec{\mathbf{e}}$ for which the inner integral on the right side of equality (1) exists and is finite, it is well known that these are almost all directions. We will deal with Sobolev spaces $W_p^l(\Omega),\ 1 \leq l < \infty$ defined as a set of the functions from the space $L_p(\Omega)$ which has in the domain Ω all generalized derivatives in the Sobolev sense up to order l inclusively and for which the next norm is finite

$$\|u\|_{W_p^l(\Omega)} = \sum_{|\beta| \leq l} \|D^\beta u\|_{L_p(\Omega)},$$

where β is a multi-index defined as $\beta = (\beta_1, \beta_2, ..., \beta_n),\ \beta_i \in \mathbb{N}_0,\ |\beta| \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \beta_i,\ D^\beta u$ are generalized partial derivatives

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}}.$$

Notation $\mathrm{Lip}\,\lambda,\ 0 < \lambda \leq 1$ means the set of functions satisfying the Holder-Lipschitz condition

$$\mathrm{Lip}\,\lambda := \left\{\rho(Q):\ |\rho(Q)-\rho(P)| \leq Mr^{\lambda},\ P,Q \in \bar{\Omega}\right\}.$$

The operator of fractional differentiation in the sense of Kipriyanov defined in [3] by formal expression

$$\mathfrak{D}^{\alpha}(Q) = \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{r}\frac{[f(Q)-f(P+\vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}}\left(\frac{t}{r}\right)^{n-1}dt + C_n^{(\alpha)}f(Q)r^{-\alpha},\ P \in \partial\Omega, \tag{2}$$

where $C_n^{(\alpha)} = (n-1)!/\Gamma(n-\alpha)$, according to theorem 2 [3] acting as follows

$$\mathfrak{D}^{\alpha}:\overset{0}{W}{}_p^l(\Omega) \to L_q(\Omega),\ lp \leq n,\ 0 < \alpha < l - \frac{n}{p} + \frac{n}{q},\ p \leq q < \frac{np}{n-lp},$$

where $\overset{0}{W}{}_p^l(\Omega)$ is a subspace of $W_p^l(\Omega)$, obtained as a result of the closure of the set of finite infinitely differentiable functions $C_0^{\infty}(\Omega)$ by the norm $W_p^l(\Omega)$. We consider $(0 < \alpha < 1)$. Using the terminology of [4], the left-side, right-side fractional derivative in the sense of Riemann-Liouville on the segment $[a,b]$ of the real axis we will denote respectively as $D_{a+}^{\alpha},\ D_{b-}^{\alpha}$; fractional derivatives in the sense of Marsho on the segment will be denoted respectively via $\mathbf{D}_{a+}^{\alpha},\ \mathbf{D}_{b-}^{\alpha}$; classes of functions representable by the fractional integral on the segment we will denote respectively by $I_{a+}^{\alpha}(L_p(a,b)),\ I_{b-}^{\alpha}(L_p(a,b)),\ 1 \leq p \leq \infty$. Everywhere further if not stated otherwise we use the notations of [2], [3], [4].

2 Main results. The following theorem establishes the strong accretive property (see [5, p. 352]) for the operator of fractional differentiation in the sense of Kipriyanov acting in the complex weight space of Lebesgue summable with squared functions.

Theorem 1. *Let $n \geq 2,\ \rho(Q)$ be a non-negative real function in class* $\mathrm{Lip}\lambda,\ \lambda > \alpha$. *Then for the operator of fractional differentiation in the sense of Kipriyanov the inequality of strong accretiveness holds*

$$\mathrm{Re}\langle f, \mathfrak{D}^{\alpha}f\rangle_{L_2(\Omega,\rho)} \geq \frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,\rho)},\ f \in \overset{0}{W}{}_2^1(\Omega). \tag{3}$$

Proof. First we assume that f is real. For $f \in C_0^{\infty}(\Omega)$ consider the following difference in which the second summand exists due to theorem 3 [2]

$$\rho(Q)f(Q)(\mathfrak{D}^{\alpha}f)(Q) - \frac{1}{2}\left(\mathfrak{D}^{\alpha}(\rho f^2)\right)(Q)$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int\limits_{0}^{r}\frac{\rho(Q)[f(P+\vec{\mathbf{e}}r)-f(P+\vec{\mathbf{e}}t)]^2}{(r-t)^{\alpha+1}}\left(\frac{t}{r}\right)^{n-1}dt + \frac{C_n^{(\alpha)}}{2}\rho(Q)|f(Q)|^2r^{-\alpha}dr \geq 0.$$

Therefore,

$$\rho(Q)f(Q)(\mathfrak{D}^{\alpha}f)(Q) \geq \frac{1}{2}\left(\mathfrak{D}^{\alpha}(\rho f^2)\right)(Q). \tag{4}$$

Integrating the left and right sides of inequality (4), then using a Fubini theorem we get the next inequality

$$\int\limits_{0}^{d(\vec{\mathbf{e}})} f(Q)(\mathfrak{D}^{\alpha} f)(Q)\rho(Q) r^{n-1} dr \geq \frac{1}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} (\mathfrak{D}^{\alpha}(\rho f^{2}))(Q) r^{n-1} dr$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int\limits_{0}^{d(\vec{\mathbf{e}})} t^{n-1} dt \int\limits_{t}^{d(\vec{\mathbf{e}})} \frac{(\rho f^{2})(Q)-(\rho f^{2})(P+\vec{\mathbf{e}}t)}{(r-t)^{\alpha+1}}\, dr + \frac{C_n^{(\alpha)}}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1-\alpha} dr$$

$$= -\frac{1}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} (\mathbf{D}^{\alpha}_{d(\vec{\mathbf{e}})-}\vartheta_{\vec{\mathbf{e}}})(Q) r^{n-1} dr + \frac{C_n^{(\alpha)}}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1-\alpha} dr$$

$$+\frac{1}{2\Gamma(1-\alpha)}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1}(d(\vec{\mathbf{e}})-r)^{-\alpha} dr = I,\ \ \vartheta_{\vec{\mathbf{e}}}(r) = (\rho f^{2})(P+r\vec{\mathbf{e}}).$$

Note that we can show, repeating the proof in Lemma 1 [6], that the sequence $\psi_{\varepsilon}(P + r\vec{\mathbf{e}})$, $\varepsilon \downarrow 0$ corresponding to a function f^2 converges in the space $L_p(\Omega_{\vec{\mathbf{e}}})\,(1 \leq p < \infty)$. In consequence of theorems: 13.2 [4, p.183], 13.7 [4, p.186] we get $\vartheta_{\vec{\mathbf{e}}} \in I^{\alpha}_{d(\vec{\mathbf{e}})-}(L_p(\Omega_{\vec{\mathbf{e}}}))$. According to theorem 13.1 [4, p.183] the next equality holds almost everywhere $\mathbf{D}^{\alpha}_{d(\vec{\mathbf{e}})-}\vartheta_{\vec{\mathbf{e}}} = D^{\alpha}_{d(\vec{\mathbf{e}})-}\vartheta_{\vec{\mathbf{e}}}$. Thus applying formula (2.64) [4, p.51] we get

$$I = -\frac{1}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} (\rho f^{2})(Q) D^{\alpha}_{0+} r^{n-1} dr$$

$$+\frac{C_n^{(\alpha)}}{2}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1-\alpha} dr + \frac{1}{2\Gamma(1-\alpha)}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1}(d(\vec{\mathbf{e}})-r)^{-\alpha} dr.$$

Using formula (2.44) [4, p. 47] we get

$$2\Gamma(1-\alpha) I = \int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1}(d(\vec{\mathbf{e}})-r)^{-\alpha} dr \geq \frac{1}{(\operatorname{diam}\Omega)^{\alpha}}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1} dr.$$

Therefore, for any direction $\vec{\mathbf{e}}$, we get the inequality

$$\int\limits_{0}^{d(\vec{\mathbf{e}})} f(Q)(\mathfrak{D}^{\alpha} f)(Q)\rho(Q) r^{n-1} dr \geq \frac{1}{2\Gamma(1-\alpha)\,(\operatorname{diam}\Omega)^{\alpha}}\int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^{2}\rho(Q) r^{n-1} dr.$$

Integrating the left and right sides of the last inequality we get

$$\langle f, \mathfrak{D}^{\alpha} f\rangle_{L_2(\Omega,\rho)} \geq \frac{1}{\lambda^{2}}\|f\|^{2}_{L_2(\Omega,\rho)},\ f \in C_0^{\infty}(\Omega),\ \lambda^{2} = 2\Gamma(1-\alpha)\,(\operatorname{diam}\Omega)^{\alpha}. \tag{5}$$

Suppose that $f \in \overset{0}{W_2^1}(\Omega)$. There is a sequence $\{f_k\} \in C_0^\infty(\Omega)$ such that $f_k \xrightarrow{W_2^1} f$. The conditions imposed on the weight function ρ imply the equivalence of norms $L_2(\Omega)$ and $L_2(\Omega,\rho)$, consequently $f_k \xrightarrow{L_2(\Omega,\rho)} f$. Using the smoothness of the weight function ρ, the embedding of spaces $L_p(\Omega)$, $p \geq 1$, and the inequality (5) of [3], we get the following estimate $\|\mathfrak{D}^\alpha f\|_{L_2(\Omega,\rho)} \leq C_1 \|\mathfrak{D}^\alpha f\|_{L_q(\Omega)} \leq C_2 \|f\|^2_{W_2^1(\Omega)}$, $2 < q < 2n/(2\alpha - 2 + n)$, $C_i > 0$, $(i = 1, 2)$. Therefore, $\mathfrak{D}^\alpha f_k \xrightarrow{L_2(\Omega,\rho)} f$. Hence from the continuity properties of the inner product in the Hilbert space, we get

$$\langle f_k, \mathfrak{D}^\alpha f_k\rangle_{L_2(\Omega,\rho)} \to \langle f, \mathfrak{D}^\alpha f\rangle_{L_2(\Omega,\rho)}.$$

Passing to the limit in the left and right sides of inequality (5) we obtain the inequality (3) in the real case. Now consider the case when f is complex-valued. Note that the following obvious equality is true

$$\mathrm{Re}\langle f, \mathfrak{D}^\alpha f\rangle_{L_2(\Omega,\rho)} = \langle u, \mathfrak{D}^\alpha u\rangle_{L_2(\Omega,\rho)} + \langle v, \mathfrak{D}^\alpha v\rangle_{L_2(\Omega,\rho)}, \tag{6}$$

$$u = \mathrm{Re} f,\ v = \mathrm{Im} f.$$

Inequality (3) follows now from equality (6). □

3 Conclusions. In this paper we proved the theorem establishing the strong accretive property for the operator of fractional differentiation in the Kipriyanov sense acting in weighted Lebesgue space of integrable squared functions.

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