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ALMOST SURJECTIVE HOMOMORPHISMS AND THEIR MEASURABILITY *

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Abstract. Some related questions concerning measurability properties of surjective homomorphisms with respect to the certain class of measures are discussed.

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1 Introduction. It is well known that measurability of functions plays a significant role in various questions of modern analysis. The measurability of functions is an important property for these functions. Moreover, measurability property of functions is frequently crucial in the process of investigation of many significantly questions of various directions of mathematics.

2 Compilation. Let (G_1, μ_1) and (G_2, μ_2) be arbitrary two uncountable groups equipped with various nonzero -finite left invariant (left quasi-invariant) measures. and let

$$f : G_1 \to G_2.$$

A function f is measurable with respect to measure μ_1 if

$$(\forall B \in dom(\mu_2)(f^{-1}(B) \in dom(\mu_1)).$$

Let M be a class of measures on G_1 (we assume, in general, that the domains of measures from M are various σ-algebras of subsets of G_1).

We shall say that a function f is relative measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is measurable with respect to μ.

Example 1. The classical Vitali theorem states the existence of a subset of the real line $\mathbf{R}$, which is nonmeasurable in the Lebesgue sense. There exist Vitali sets on $\mathbf{R}$ which are relatively measurable with respect to the class of all those measures on $\mathbf{R}$ which extend λ and are translation quasi-invariant.

About example 1 see [1].

Let μ be a measure on G such that

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$$(\forall x)(x \in G \Rightarrow \{x\} \in dom(\mu)).$$

As usual we say that μ is diffused if $\mu(\{x\}) = 0$ for each $x \in G$.

Let μ be a σ-finite measure given on a base set G. A subset X of G is called μ-thick if $\mu_*(E \setminus X) = 0$, where μ_* denotes the inner measure associated with μ n.

Notice that, the thickness of graphs is the pathological property for subsets of the basic set. However, this phenomenon plays an essential role in the problem of extensions of measures.

Example 2. In [2] it was demonstrated that there exists an additive function

$$f : \mathbf{R} \to \mathbf{R}$$

having the following property: for any σ-finite diffused Borel measure μ on $\mathbf{R}$ and for any σ-finite measure ν on $\mathbf{R}$, the graph of f is a $(\mu \times \nu)$-thick subset of the Euclidean plane $\mathbf{R}^2$. In this case an additive function

$$f : \mathbf{R} \to \mathbf{R}$$

is relatively measurable with respect to the class of all extensions of sigma-finite diffused Borel measures on $\mathbf{R}$.

Example 3. Let M_1 be the class of all nonzero σ-finite separable measures on $\mathbf{R}^\mathbf{N}$ and let M_2 be the class of all nonzero σ-finite non-separable measures on $\mathbf{R}^\mathbf{N}$. If a function $f : E \to \mathbf{R}$ is relatively measurable with respect to the class M_2, then f is relatively measurable with respect to the class M_1 (see [3]).

The following lemma is valid.

Proposition 1. *Let the group G_2 be equipped with a σ-finite G_2-left-invariant (left G_2-quasi-invariant) measure μ_2. Consider the family of sets $S = \{f^{-1}(Y) : Y \in dom(\mu)\}$, and define a functional μ_1 on this family by putting*

$$\mu_1(\varphi^{-1}(Y)) = \mu_2(Y),$$

where $Y \in dom(\mu_2)$.

Then this functional is a measure satisfying the following relations:

(a) S is a G_1-left-invariant σ-algebra of subsets of G_1;

(b) μ_2 is a non-atomic σ-finite G_1-left-invariant measure on S.

The proof of Proposition 1 is presented in [4].

Let

$$f : G_1 \to G_2$$

be a homomorphism. We say that f is an almost surjective homomorphism if the graph of f is $(\mu_1 \times \mu_2)$-thick in $G_1 \times G_2$.

Our argument may be regarded as a certain combination of the method of Kodaira and Kakutani with the method of surjective homomorphisms.

The following theorems are true.

Theorem 1. *Let G_2 be a group equipped with a G_2-left-invariant probability measure and let $f : G_1 \to G_2$ be a almost surjective homomorphism. Then there exists a G_1-left-invariant measure μ_1' on G_1 which is extension of measure μ_1 and f is measurable with respect to μ_1'.*

Proof. Suppose that for sets $Y_1 \in dom(\mu_2)$ and $Y_2 \in dom(\mu_2)$ the following assertion is true: $f^{-1}(Y_1) = f^{-1}(Y_2)$. Consequently, we have $f^{-1}(Y_1 \triangle Y_2) = \emptyset$. Therefore, we get $(Y_1 \triangle Y_2) \cap Gr(f) = \emptyset$.

In view of the thickness of the graph $Gr(f)$ of f, we infer that

$$\mu_2(Y_1 \triangle Y_2) = 0.$$

Hence $\mu_2(Y_1) = \mu_2(Y_2)$.

This implies that the definition of μ_1 is correct. In a similar manner of Proposition 1, we prove the left-invariance of measure μ_1 under the group G_1.

Now, For each $(\mu_1 \times \mu_2)$-measurable set $Z \subset G_1 \times G_2$, we denote

$$Z' = \{x \in G_1 : (x, f(x)) \in Z\}.$$

Further, we put

$$S = \{Z' : Z \in dom(\mu_1 \times \mu_2)\}.$$

It can easily be verified that S is a sigma-algebra of subsets of $G_1 \times G_2$. We define a functional μ_1' on S by the formula

$$\mu_1'(Z') = (\mu_1 \times \mu_2)(Z) \qquad (Z \in dom(\mu_1 \times \mu_2)).$$

It is easy to show that the definition of μ_1' is correct in view of the $(\mu_1 \times \mu_2)$-thickness of the graph of f. Also, μ_1' turns out to be a measure on S, which extends the original measure μ_1.

Let $B \in dom(\mu_2)$ and let $x \in f^{-1}(B)$. Then we have

$$f(x) \in B);$$

$$(x, f(x)) \in G_1 \times B \in dom(\mu_1 \times \mu_2);$$

$$S = \{x, f(x) \in G_1 \times B\} = \{x : x \in f^{-1}(B)\};$$

Finally, we get

$$(\forall B)(B \in dom(\mu_2 \Rightarrow f^{-1}(B) \in dom(\mu_1).$$

This ends the proof of Theorem 2. □

From Theorem 1 there follows the next proposition.

Theorem 2. *Let G_2 be a group equipped with a G_2-left-invariant σ-finite measure and let $f : G_1 \to G_2$ be an almost surjective homomorphism.*

Then there exists a G_1-left-quasi-invariant measure μ_1' on G_1 which is the extension of measure μ_1 and f is relatively measurable with respect to μ_1'.

3 Conclusions. In the present paper, an approach to some questions of measurability of functions is discussed, which is rather useful in certain situations and present several statements, which are generalizations of the classical definitions of measurability of functions from a certain point of view.

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