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FINITE DIFFERENCE SCHEME FOR ONE SYSTEM OF NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH SOURCE TERMS *

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Abstract. Finite difference scheme for one system of nonlinear partial integro-differential equations with source terms is investigated. Mentioned system based on Maxwell equations describing the process of propagation of the electromagnetic field into a substance. Large time behavior of solution of corresponding initial-boundary value problem and finite difference scheme are studied. Wider class of nonlinearity is studied than one has been investigated before.

Keywords and phrases: System of nonlinear partial integro-differential equations, source term, initial-boundary value problem, asymptotic behavior, finite difference scheme, convergence.

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Mathematical models of diffusive processes lead to non-stationary partial differential and integro-differential equations and systems of those equations. Most of those problems, as a rule are nonlinear. This moment significantly complicates the investigation of such models.

Purpose of this note is the investigation and numerical resolution of such nonlinear diffusion system, which appears at mathematical modeling of electromagnetic field propagation into a medium [5]. The main characteristic of that system besides of nonlinearity is that they contain equations, which are strongly connected to each other. This circumstance dictates to use the corresponding investigation methods for each concrete model, as the general theory even for such linear systems is not yet fully developed. Naturally, the questions of numerical solution of those problems, which also are connected with serious complexities, arise as well.

In particular, our aim is to study the following system of nonlinear integro-differential equations with source terms:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a\left(S\right)\frac{\partial U}{\partial x}\right] + |U|^{q-2}U = 0, \quad \frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left[a\left(S\right)\frac{\partial V}{\partial x}\right] + |V|^{q-2}V = 0, \tag{1}$$

where

$$S(x,t) = \int\limits_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau,$$

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$a(S) = (1+S)^p$, $0 < p \leq 1$; $q \geq 2$.

The mentioned system (1) is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell equations [5] to the system of nonlinear integro-differential equations. Such a reduction at first was made in [4]. The investigations of qualitative and quantitative properties of solutions of the initial-boundary value problems for such of systems are studied in the following works [1] - [3], [6] - [8].

In the domain $[0,1]\times[0,\infty)$ let us consider the following initial-boundary value problem with homogeneous mixed boundary conditions:

$$\begin{gathered}
\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U}{\partial x}\right\} + |U|^{q-2}U = 0,\\
\frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V}{\partial x}\right\} + |V|^{q-2}V = 0,
\end{gathered} \tag{2}$$

$$U(0,t) = V(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1} = 0, \quad t \geq 0, \tag{3}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{4}$$

where $0 < p \leq 1$, $q \geq 2$.

The following statement takes place.

Theorem 1. *If $0 < p \leq 1$, $q \geq 2$ and $U_0(0) = V_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = \left.\frac{dV_0(x)}{dx}\right|_{x=1} = 0$, $U_0, V_0 \in W_2^1(0,1)$, then for the solution of problem (2) - (4) the following asymptotic estimate is true*

$$\|U\|_{L_2(0,1)} + \|V\|_{L_2(0,1)} \leq C\exp(-t).$$

Now in the domain $Q_T = (0,1)\times(0,T)$, where T is a positive constant, let us consider the finite difference scheme for the following nonlinear integro-differential problem:

$$\begin{gathered}
\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U}{\partial x}\right\} + |U|^{q-2}U = f_1(x,t),\\
\frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V}{\partial x}\right\} + |V|^{q-2}V = f_2(x,t),\\
U(0,t) = V(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1} = 0, \quad t \geq 0,\\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x).
\end{gathered} \tag{5}$$

Here $0 < p \le 1$; $q \ge 2$, $f_1 = f_1(x,t)$, $f_2 = f_2(x,t)$, $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions of their arguments.

Our aim is to investigate the finite difference scheme for wider classes of nonlinearity than one that has been already studied.

Let us correspond to problem (5) the following finite difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau} - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p u_{\bar{x},i}^{j+1} \right\}_x + |u_i^{j+1}|^{q-2} u_i^{j+1} = f_{1,i}^j,$$

$$\frac{v_i^{j+1}-v_i^j}{\tau} - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p v_{\bar{x},i}^{j+1} \right\}_x + |v_i^{j+1}|^{q-2} v_i^{j+1} = f_{2,i}^j, \tag{6}$$

$$i = 1,2,...,M-1; \quad j = 0,1,...,N-1,$$

$$u_0^j = v_0^j = u_{\bar{x},M}^j = v_{\bar{x},M}^j = 0, \quad j = 0,1...,N,$$

$$u_i^0 = U_{0,i}, \ , v_i^0 = V_{0,i}, \quad i = 0,1,...,M.$$

Here we use the following known notations [9]:

$$r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}, \quad r_{\bar{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}.$$

Multiplying equations in (6) scalarly by u_i^{j+1} and v_i^{j+1} respectively, it is not difficult to get the inequalities which guarantee the stability of the scheme (6):

$$\|u^n\|^2 + \sum_{j=1}^n \|u_{\bar{x}}^j]|^2 \tau < C, \quad \|v^n\|^2 + \sum_{j=1}^n \|v_{\bar{x}}^j]|^2 \tau < C, \quad n = 1,2,...,N,$$

where here and below C is a positive constant independent from τ and h.

The following convergence theorem takes place

Theorem 2. *If problem (5) has a sufficiently smooth solution $(U(x,t),\ V(x,t))$, then the solution $u^j = (u_1^j, u_2^j, \ldots, u_M^j)$, $v^j = (v_1^j, v_2^j, \ldots, v_M^j)$, $j = 1,2,\ldots,N$ of the difference scheme (6) tends to the solution of the continuous problem (5) $U^j = (U_1^j, U_2^j, \ldots, U_M^j)$, $V^j = (V_1^j, V_2^j, \ldots, V_M^j)$, $j = 1,2,\ldots,N$ as $\tau \to 0$, $h \to 0$ and the following estimates are true:*

$$\|u^j - U^j\| \le C(\tau + h), \quad \|v^j - V^j\| \le C(\tau + h).$$

Note, that applying the technique of proving convergence theorems, it is not difficult to prove the uniqueness of the solutions of problem (2) - (4) as well as of the scheme (6).

Various numerical experiments are carried out. Results of numerical experiments fully agree with theoretical researches in both, in convergence of the finite discrete scheme as well as in asymptotic behavior of solution.

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