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GALERKIN APPROXIMATION OF THE SOLUTION OF A NONLINEAR BEAM EQUATION

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Abstract. An initial boundary value problem is posed for the Timoshenko type nonlinear integro-differential inhomogeneous equation which describes the dynamic behaviour of a beam. To approximate the solution with respect to a spatial variable the Galerkin method is used the error of which is estimated.

Keywords and phrases: Beam equation, Galerkin method, error estimate.

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1 Problem formulation.

Let us consider the nonlinear differential equation

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-h\frac{\partial^4 u}{\partial x^2\partial t^2}(x,t)\\ -\left(\lambda+\int_0^L\left(\frac{\partial u}{\partial \xi}(\xi t)\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x}^2(x,t)=f(x,t),\\ 0<x<L,\quad 0<t\leq T,\end{gathered}\tag{1}$$

with the initial boundary equations

$$\begin{gathered}u\,(x,0)=u^0\,(x)\,,\qquad \frac{\partial u}{\partial t}\,(x,0)=u^1\,(x)\,,\\ u\,(0,t)=u\,(L,t)=0,\qquad \frac{\partial^2 u}{\partial x^2}\,(0,t)=\frac{\partial^2 u}{\partial x^2}\,(L,t)=0,\\ 0\leq x\leq L,\quad 0\leq t\leq T,\end{gathered}\tag{2}$$

where $h\geq 0$ and $\lambda>0$ are some given constants, $u^0(x)$, $u^1(x)$ and $f(x,t)$ are the given functions and $u\,(x,t)$ is the function we want to obtain. Equation (1) describes the oscillation of a beam. It is given in E. Henriques de Brito [2] and belongs to the class of equations based on the Timoshenko theory. For $\lambda=0$ (1) is derived in [3] by passing to the limit in the one-dimensional version of the von Karman system describing approxemately the plane motion of a uniform prismatic beam.

Here we consider one of the questions of approximate algoritms for equation (1). Numerical me-thods for nonlinear integro-differential beam equations are investigated in [1, 4, 5].

2 Assumptions. Suppose that the initial functions $u^0(x)$, $u^1(x)$ and the function $f(x,t)$ are represented in the form $u^l(x) = \sum_{i=1}^{\infty} u_i^{(l)} sin\frac{i\pi x}{L}$, $l = 0,1$, $f(x,t) = \sum_{i=1}^{\infty} f_i(t) sin\frac{i\pi x}{L}$ and

$$u_i^{(l)^2} \le \frac{\omega_l}{i^{p_l+5}}, \qquad l = 0,1, \qquad f_i^2(t) \le \frac{\omega}{i^{p+1}}, \tag{3}$$

where p_l, ω_l, $l = 0,1$, and p , ω are some positive numbers.

Suppose that there exists a solution of problem (1), (2) which is represented in the form $u(x,t) = \sum_{i=1}^{\infty} u_i(t)\, sin\frac{i\pi x}{L}$, where the coefficients $u_i(t)$ satisfy the following infinite system of differential equations

$$\left(1 + h\left(\frac{\pi i}{L}\right)^2\right) u_i''(t) + \left(\frac{\pi i}{L}\right)^4 u_i(t) + \left(\lambda + \frac{L}{2}\sum_{j=1}^{\infty}\left(\frac{\pi j}{L}\right)^2 u_j^2(t)\right)\left(\frac{\pi i}{L}\right)^2 u_i(t) = f_i(t),$$

$$i = 1,\ 2, ..., \qquad 0 < t \le T,$$

with the initial conditions

$$u_i(0) = u_i^{(0)}, \qquad u_i'(0) = u_i^{(1)}, \qquad i = 1,\ 2, ...,$$

where $f_i(t) = \frac{2}{L}\int_0^L f(x,\ t)\sin\frac{i\pi x}{L}\,dx$. Finally, we assume that the series $\sum_{i=1}^{\infty} i^2 u_i'^2(t)$ and $\sum_{i=1}^{\infty} i^4 u_i^2(t)$ converge.

Note that in [2], the existence and uniqueness of a weak solution is proved for the equation a particular case of which is equation (1).

3 Galerkin method. An approximate solution of problem (1), (2) is written in the form

$$u_n(x,\ t) = \sum_{i=1}^{n} u_{ni}(t)\sin\frac{i\pi x}{L},$$

where the coefficients $u_{ni}(t)$ are defined by the Galerkin method from the system of nonlinear differential equations

$$\left(1 + h\left(\frac{\pi i}{L}\right)^2\right) u_{ni}''(t) + \left(\frac{\pi i}{L}\right)^4 u_{ni}(t) + \left(\lambda + \frac{L}{2}\sum_{j=1}^{n}\left(\frac{\pi j}{L}\right)^2 u_{nj}^2(t)\right)\left(\frac{\pi i}{L}\right)^2 u_{ni}(t) = f_i(t),$$

$$i = 1,\ 2, ..., n, \quad 0 < t \le T,$$

and the conditions

$$u_{ni}(0) = u_i^{(0)}, \qquad u_{ni}'(0) = u_i^{(1)}, \qquad i = 1,\ 2, ..., n.$$

4 Method error. Under the method error we will undenstand the difference between the approximate and exact solutions $u_n(x,\ t) - u(x,\ t)$ the $L^2(0,\ L)$ norm $\|\|$ of which we want to estimate. Let us formulate the main result.

Theorem 1. *The inequality*

$$\|(u_n(x,t) - u(x,t))_t\|^2 + h\|(u_n(x,t) - u(x,t))_{xt}\|^2 + \|(u_n(x,t) - u(x,t))_{xx}\|^2 +$$

$$+ \lambda\|(u_n(x,t) - u(x,t))_x\|^2 \leq r_n(t) + \frac{1}{2}\left(\int_0^T \sigma_1(t)\, r_n^2(t)\, dt\right) exp\left(\int_0^t \sigma_2(\tau)\, d\tau\right) \quad (4)$$

s fulfilled for the error of the Galerkin method.

Here

$$r_n = 2\left(\psi_n + \frac{\alpha_2^2}{\alpha_0 {\alpha^2}_1}\int_0^t \frac{1}{c_1^2(\tau)}\|\Delta_n f(x,\tau)\|^2 d\tau\right) exp\left(\alpha_0 {\alpha^2}_1 \int_0^t c_1^2(\tau) d\tau\right),$$

$$\psi_n = \left\|\Delta_n u^1(x)\right\|^2 + \left\|\Delta_n u^{0''}(x)\right\|^2 + h\left\|\Delta_n u^{1'}(x)\right\|^2 + \left(\lambda + \left\|u^{0'}(x)\right\|^2\right)\left\|\Delta_n u^{0'}(x)\right\|^2,$$

$$\sigma_1(t) = \frac{1}{2\alpha_0}{\alpha^2}_1 \alpha_2^2 \frac{c_1^2(t)}{c_1^2(t) + c_3^2(t)}\ ,\ \sigma_2(t) = \frac{1}{2}\alpha_1\left(\alpha_0\alpha_1\left(c_1^2(t) + c_3^2(t)\right)\right.$$
$$\left. + \alpha_2\left(c_0(t) + c_2(t)\right)\left(c_1(t) + c_3(t)\right)\right),$$

$$\Delta_n u^l(x) = \sum_{i=n+1}^{\infty} u_i^{(l)} sin\frac{i\pi x}{L}\ , \qquad l = 0,\ 1, \qquad \Delta_n f(x,t) = \sum_{i=n+1}^{\infty} f_i(t) sin\frac{i\pi x}{L}\ ,$$

$$\alpha_0 = \frac{1}{1 + (h\lambda)^{\frac{1}{2}}}\ , \qquad \alpha_1 = \left(\frac{1}{1 + \lambda\left(\frac{L}{\pi}\right)^2}\right)^{\frac{1}{2}} \frac{L}{\pi}, \qquad \alpha_2 = \left(\frac{1}{1 + h\left(\frac{\pi}{L}\right)^2}\right)^{\frac{1}{2}},$$

$$c_0(t) = \left(\left(\left(\frac{\pi}{L}\right)^4 + 2\lambda\left(\frac{\pi}{L}\right)^2 + 2ke^t\right)^{\frac{1}{2}} - \left(\frac{\pi}{L}\right)^2 - \lambda\right)^{\frac{1}{2}}, \quad c_1(t) = \left(ke^t\right)^{\frac{1}{2}}\ ,$$

$$c_2(t) = \left(\left(\left(\frac{\pi}{L}\right)^4 + 2\lambda\left(\frac{\pi}{L}\right)^2 + 2k_n e^t\right)^{\frac{1}{2}} - \left(\frac{\pi}{L}\right)^2 - \lambda\right)^{\frac{1}{2}}, \quad c_3(t) = \left(k_n e^t\right)^{\frac{1}{2}},$$

$$k = \|u^1(x)\|^2 + \left\|u^{0''}(x)\right\|^2 + h\left\|u^{1'}(x)\right\|^2 + \tfrac{1}{2}\left(\lambda + \left\|u^{0'}(x)\right\|^2\right)^2 + \alpha_2^2 \int_0^T \|f(x,\ t)\|^2 dt,$$

$$k_n = \|\pi_n u^1(x)\|^2 + \left\|\pi_n u^{0''}(x)\right\|^2 + h\left\|\pi_n u^{1'}(x)\right\|^2$$

$$+\frac{1}{2}\left(\lambda + \left\|\pi_n u^{0'}(x)\right\|^2\right)^2 + \alpha_2^2 \int_0^T \|\pi_n f(x,\ t)\|^2 dt,$$

$$\pi_n u^l(x) = \sum_{i=1}^n u_i^{(l)} sin\frac{i\pi x}{L}, \quad l = 0,\ 1, \quad \pi_n f(x,t) = \sum_{i=1}^n f_i(t) sin\frac{i\pi x}{L}\ .$$

Using (3) it is possible to derive formulas, which allow to estimate right side of inequality (3) by parameters $p_l,\ \omega_l,\quad l = 0,\ 1,\ \ p,\ \ \omega$ and n. The same formulas can be also obtained in case, when $u_i^{(l)},\ \ l = 0, 1,$ and $f_i(t)$ change by a rule different from (3).

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