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THE LEGENDRE FUNCTIONS OF THE SECOND KIND. NEW RELATIONS *

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Abstract. We propose the new connection between the Legendre functions of the second kind. It is shown that the Legendre associated function of the second kind is analytic when $\mathrm{Re}\mu = \mathrm{Im}\nu = 0$, where μ is the superscript and ν is the lower index of the function, respectively. Besides, we have gotten new asymptotic relations.

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1 Introduction. We have obtained the connection between the Legendre functions of the second kind $Q_\nu^\mu(z)$ and $Q_\nu(z)$ when $\mathrm{Re}\mu = \mathrm{Im}\nu = 0$. Also, we have derived asymptotic expressions of the function $Q_\nu^{i\tau}(z)$ for the case $|\nu| \to \infty$ and have found its behavior in the neighborhood of the point $z = 1$, when $\mathrm{Re}\mu = \mathrm{Im}\nu = 0$.

2 Content. To investigate the functions mentioned above, the next integral formula (see, e. g. [1], p. 960) is used:

$$Q_\nu^\mu(z) = \exp(i\pi\mu)\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty dt\cosh(\mu t)\left(z+\sqrt{z^2-1}\cosh(t)\right)^{-\nu-1}, \tag{1}$$

$$\mathrm{Re}(\nu+\mu) > -1, \ \ \nu \neq -1, \ -2, \ -3, \ . \ . \ . \ , \ \ |\arg(z\pm 1)| < \pi.$$

When $\mu = i\tau$, one gets:

$$Q_\nu^{i\tau}(z) = \exp(-\pi\tau)\frac{\Gamma(\nu+1)}{\Gamma(\nu-i\tau+1)}\int_0^\infty dt\cos(\tau t)\left(z+\sqrt{z^2-1}\cosh(t)\right)^{-\nu-1}, \tag{2}$$

$$\mathrm{Re}(\nu) > -1, \quad |\arg(z\pm 1)| < \pi,$$

and for $\mu = 0$ one obtains the Legendre function of the second kind:

$$Q_\nu(z) = \int_0^\infty dt\Big(z+\sqrt{z^2-1}\cos h(t)\Big)^{-\nu-1}. \tag{3}$$

Let us consider the case $z = a$, where a is real and $a > 1$. Then the relation holds:

$$\Big(a+\sqrt{a^2-1}\cosh(t)\Big)^{-\nu-1} = 0, \quad t\to\infty, \quad \mathrm{Re}(\nu) > -1. \tag{4}$$

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Therefore, from (2) one gets:

$$Q_{\nu}^{i\tau}(a) = \exp(-\pi\tau)\frac{\Gamma(\nu+1)}{\Gamma(\nu-i\tau+1)}\int_0^A dt\cos(\tau t)\left(a+\sqrt{a^2-1}\cosh(t)\right)^{-\nu-1} + O(A), \quad (5)$$

where $A<\infty$. Similarly, for representation (3) we have:

$$Q_{\nu}(a) = \int_0^A dt\left(a+\sqrt{a^2-1}\cos h(t)\right)^{-\nu-1} + O(A). \quad (6)$$

Note also, that function (4) is the impulse function and

$$\left(a+\sqrt{a^2-1}\cosh(t)\right)^{-\nu-1} \geq 0, \quad a>1.$$

Hence, in the integral formula (5) one can use the first theorem of the mean:

$$Q_{\nu}^{i\tau}(a) = \exp(-\pi\tau)\frac{\Gamma(\nu+1)}{\Gamma(\nu-i\tau+1)}\cos(\tau\eta)\int_0^A dt\left(a+\sqrt{a^2-1}\cosh(t)\right)^{-\nu-1} + O(A), \quad (7)$$
$$0\leq\eta\leq A, \quad \mathrm{Re}\nu>-1.$$

Inserting (6) into formula (7) one gets the next connection:

$$Q_{\nu}^{i\tau}(a) = \exp(-\pi\tau)\frac{\Gamma(\nu+1)}{\Gamma(\nu-i\tau+1)}\cos(\tau\eta)\,Q_{\nu}(a), \quad a>1, \quad (8)$$
$$0\leq\eta\leq A, \quad \mathrm{Re}\nu>-1.$$

Since both functions $Q_{\nu}^{\mu}(z)$ and $Q_{\nu}(z)$ in the formula (8) are analytic in a finite region of the complex plane with a cut along the real axis $(-\infty, 1]$ ([1], p. 959, [2], p. 169), one can use the analytic continuation:

$$Q_{\nu}^{i\tau}(z) = \exp(-\pi\tau)\frac{\Gamma(\nu+1)}{\Gamma(\nu-i\tau+1)}\cos(\tau\eta)\,Q_{\nu}(z), \quad (9)$$
$$0\leq\eta\leq A, \quad \mathrm{Re}\nu>-1, \quad |\arg(z-1)|<\pi.$$

In order to find the real parameter η one has to rewrite (9) in the area $|z|\to\infty$. Inserting the asymptotic relation (see, e. g. [3], p. 165)

$$Q_{\nu}^{\mu}(z) = \sqrt{\pi}\exp(i\pi\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}(2z)^{-\nu-1}, \quad z\to\infty \quad (10)$$

in both sides of the formula (9), after simplifications we get a specification of the parameter η as

$$\cos(\tau\eta) = \frac{\Gamma(\nu - i\tau + 1)\,\Gamma(\nu + i\tau + 1)}{\Gamma^2(\nu + 1)}. \tag{11}$$

The equation (11) has real solutions for the parameter η if $\mathrm{Im}(\nu) = 0$. In this case according to the known relation (see, e. g. [4], p. 82)

$$|\Gamma(\nu - i\tau + 1)| \le |\Gamma(\nu + 1)|,$$

one obtains from (11) $|\cos(\tau\eta)| \le 1$. Inserting (11) into formula (9) one can write:

$$Q_\nu^{i\tau}(z) = \exp(-\pi\tau)\frac{\Gamma(\nu + i\tau + 1)}{\Gamma(\nu + 1)} Q_\nu(z), \tag{12}$$
$$\mathrm{Im}(\nu) = 0, \quad \mathrm{Re}(\nu) > -1, \quad |\arg(z-1)| < \pi.$$

So, we have shown, that for $\mathrm{Re}(\mu) = 0$, $\mathrm{Im}(\nu) = 0$ and $\mathrm{Re}(\nu) > -1$ the behavior of $Q_\nu^\mu(z)$ in the area $|\arg(z-1)| < \pi$ is determined by the function $Q_\nu(z)$.

Due to the mentioned above and according to the formula (12), the principal value of $Q_\nu^\mu(z)$ is analytic if z does not lie on the cut: $(-\infty, 1]$. The function $Q_\nu(z)$ is also analytic with respect to ν (see, e. g. [5], p. 157). Obviously, the same property has $Q_\nu^{i\tau}(z)$.

When $|\nu| \to \infty$, due to the asymptotic behavior (see, e. g. [3], p. 62)

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}, \quad |z| \to \infty, \tag{13}$$

one gets from (11)

$$\cos(\tau\eta) = 1, \;\; \eta = 2\pi k/\tau, \;\; k = 0, \pm 1, ..., \quad \tau \ne 0$$

In this case, inserting (11) in the formula (9) and using formula (13), after simplifications an asymptotic expression one obtains:

$$Q_\nu^{i\tau}(z) = \exp(-\pi\tau)\nu^{i\tau} Q_\nu(z), \quad \mathrm{Re}\nu > -1, \quad |\nu| \to \infty. \tag{14}$$

Besides, inserting into (12) the asymptotic relation (see, e. g. [6], p. 220):

$$Q_\nu(z) = -\frac{\ln(z-1)}{2\Gamma(\nu+1)}, \quad z \to 1, \tag{15}$$
$$\nu \ne -1, \;\; -2, \;\; -3, \;\; . \;\; . \;\; . \;\; , \;\; |\arg(z-1)| < \pi,$$

one can find the behavior of the function $Q_\nu^\mu(z)$ in the neighborhood of the point $z = 1$:

$$Q_\nu^{i\tau}(z) = -\frac{1}{2}\exp(-\pi\tau)\frac{\Gamma(\nu + i\tau + 1)}{\Gamma^2(\nu + 1)} \ln(z-1), \quad z \to 1, \tag{16}$$
$$\mathrm{Im}\nu = 0, \quad \mathrm{Re}\nu > -1, \quad |\arg(z-1)| < \pi.$$

3 Conclusions. We have gotten the connection between the Legendre functions of the second kind (see the formula (12)).

We have derived an asymptotic relation (14) between the functions $Q_{\nu}^{i\tau}(z)$ and $Q_{\nu}(z)$ for $\mathrm{Re}\nu > -1, \quad |\nu| \to \infty$. We have shown the behavior (16) of the function $Q_{\nu}^{i\tau}(z)$ in the neighborhood of the point $z = 1$.

The results obtained are useful in avoiding some quantum mechanical problems, for instance, in calculations of a non-relativistic transition amplitude of two charged particles of continuous spectra in massive photon approximation.

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