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WELL-POSEDNESS AND APPROXIMATE SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OBTAINED BY THE REDUCTION OF MAXWELL SYSTEM

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Abstract. Nonlinear parabolic integro-differential model obtained by the reduction of well-known Maxwell system of partial differential equations is considered. Unique solvability of the stated initial-boundary value problem and asymptotic behavior of solution as $t \to \infty$ are investigated. The semi-discrete and implicit finite-difference schemes are constructed. Stability and convergence of those schemes are given.

Keywords and phrases: Nonlinear parabolic integro-differential equation, initial-boundary value problem, unique solvability, asymptotic behavior, semi-discrete and finite difference schemes, stability, convergence.

AMS subject classification (2010): 45K05, 65N06.

The nonlinear integro-differential equations and their systems describe various processes in physics, economics, chemistry, technology and so on. It is doubtless that study of qualitative and structural properties of the solutions of initial-boundary problems for these models, construction and investigation of discrete analogues are very important. There are many authors who investigate models of this kind (see, for example [1], [3], [5], [13], [19] and references therein). Integro-differential models of this kind arise for example in the mathematical modeling of the process of a magnetic field penetration into a substance. In a quasi-stationary case the corresponding system of Maxwell equations [14] can be rewritten in the nonlinear integro-differential form [4], one-dimensional scalar case of which has the following form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right] \tag{1}$$

where $a = a(S)$ is defined for $S \in [0, \infty)$.

G. Laptev in his doctoral dissertation proposed some generalization of the models obtained in [4]. In one-dimensional scalar case his so called averaged integro-differential model has the following form

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2}. \tag{2}$$

Note that the integro-differential models (1) and (2) are complex. Equations and systems of such types still yield the investigation for special cases. Such models are object of investigation of many scientists see, for example, [2], [4], [6]-[12], [15]-[18], [20], [21] and references therein. The existence and uniqueness of solutions of the initial-boundary value problems for the equations of type (1) and (2) are studied in the following works [2], [4], [6]-[12], [15]-[18], [20], [21] and etc. Large time behavior of solutions for (1) and (2) type models are studied, for example, in [7]-[12], [20] and in a number of other works as well. Numerical approximation of (1) and (2) type models are studied in [6]-[12], [15]-[18], [20]. In these directions the latest and rather complete bibliography can be found in the following monograph [11].

It should be noticed that, asymptotic properties of solution and numerical solution for equation (2) for the case $a(S) = 1 + S$ are studied in [10].

Our aim is to continue study of the first type initial-boundary value problem for the equation (2). Attention will be paid to the following case: $a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$. One must note that the analogical questions for that kind of general nonlinearity above for (2) equation are studied in [9].

In the domain $[0,1] \times [0,\infty)$ for equation (2) let us consider the problem:

$$\begin{gathered} U(0,t) = U(1,t) = 0, \\ U(x,0) = U_0(x). \end{gathered} \tag{3}$$

Theorem 1. *If $a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$ and $U_0 \in H^2(0,1) \cap H_0^1(0,1)$, then there exists the unique solution U of the problem (2),(3) such that: $U \in L_2(0,\infty; H^2(0,1))$, $U_{xt} \in L_2(0,\infty; L_2(0,1))$ and*

$$\left\| \frac{\partial U}{\partial x} \right\| + \left\| \frac{\partial U}{\partial t} \right\| \leq C \exp\left(-\frac{a_0 t}{2} \right).$$

Here $H^2(0,1)$, $H_0^1(0,1)$ are Sobolev spaces, $\|\cdot\|$ is the norm of the space $L_2(0,1)$ and C is a positive constant independent from t.

In the rectangle $[0,1] \times [0,T]$, where T is a positive constant, using known notations let us construct the following semi-discrete scheme:

$$\begin{gathered} \frac{du_i}{dt} - a\left(h \sum_{l=1}^{M} \int_0^t (u_{\bar{x},l})^2 d\tau \right) u_{\bar{x}x,i} = 0, \\ i = 1,2,...,M-1, \\ u_0(t) = u_M(t) = 0, \\ u_i(0) = U_{0,i}, \quad i = 0,1,...,M. \end{gathered} \tag{4}$$

It is not difficult to get the following estimation for the solution of problem (4)

$$\|u(t)\|^2 + \int_0^t \|u_{\bar{x}}]|^2 d\tau \leq C, \tag{5}$$

where $\|\cdot\|$ and $\|\cdot]|$ are the discrete analogs of the norm of the space $L_2(0,1)$ and C is a positive constant independent from h. The a priori estimate (5) guarantees the stability and unique solvability of the semi-discrete scheme (4).

Theorem 2. *If $a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$ and problem (2),(3) has the sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t)$ of the scheme (4) tends to the solution $U = U(t)$ of the continuous problem as $h \to 0$ and*

$$\|u(t) - U(t)\| = O(h).$$

Let us construct the following implicit finite-difference scheme:

$$\begin{gathered} u^j_{t,i} - a\left(\tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u^k_{\bar{x},l})^2\right) u_{\bar{x}x,i} = 0, \\ i = 1,2,...,M-1, \quad j = 0,1,...,N-1, \\ u^j_0 = u^j_M = 0, \quad j = 0,1,...,N, \\ u^0_i = U_{0,i}, \quad i = 0,1,...,M. \end{gathered} \tag{6}$$

It is not difficult to get the following inequality which guarantees the stability and solvability of the scheme (6):

$$\|u^n\|^2 + \sum_{j=1}^{n} \|u^j_{\bar{x}}]|^2 \tau \leq C, \quad n = 1,2,...,N.$$

Here and below C is a positive constant independent from τ and h. The uniqueness of the solution of the scheme (6) are also proved.

Theorem 3. *If $a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$ and the problem (2),(3) has the sufficiently smooth solution $U = U(x,t)$, then the solution $u^j = (u^j_1, u^j_2, ..., u^j_{M-1})$, $j = 1,2,...,N$, of the difference scheme (6) tends to the solution $U^j = (U^j_1, U^j_2, ..., U^j_{M-1})$, $j = 1,2,...,N$ of the continuous problem (2),(3) as $\tau \to 0$, $h \to 0$ and*

$$\|u^j - U^j\| = O(\tau + h).$$

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