BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY OF POROUS COSSEERAT MEDIA FOR SOLIDS WITH TRIPLE-POROSITY

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Abstract. The purpose of this paper is to consider the two-dimensional version of the linear theory of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium. Using the analytic functions of a complex variable and solutions of the Helmholtz equation basic boundary value problems are solved explicitly for the circle.

Keywords and phrases: Triple-porosity, the stress tensor, a circle.

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1 The plane deformation. Basic equations. In this paper we consider the two-dimensional version of the linear theory of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium [1-4].

Let $D$ be a circle with the radius $R$. Let us assume that the domain $D$ is filled with an isotropic material with triple-porosity [5, 6]. The basic homogeneous system of equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows

$$\partial_\alpha \sigma_{\alpha\beta} = 0, \quad \partial_\alpha \mu_{\alpha3} + (\sigma_{12} - \sigma_{21}) = 0, \quad (\alpha, \beta = 1, 2)$$

$$\sigma_{\alpha\alpha} = -\beta_i p_i + \lambda \partial_\alpha u_\alpha, \quad \sigma_{12} = (\mu + \alpha) \partial_1 u_2 + (\mu - \alpha) \partial_2 u_1 - 2\alpha \omega,$$

$$\sigma_{21} = (\mu + \alpha) \partial_2 u_1 + (\mu - \alpha) \partial_1 u_2 + 2\alpha \omega, \quad \mu_{\alpha3} = (\nu + \beta) \partial_\alpha \omega, \quad \theta := \partial_1 u_1 + \partial_2 u_2,$$

where $\sigma_{\alpha\beta}$ are stress tensor components, $\mu_{\alpha3}$ are moment stress tensor components, $u_\alpha$ are components of the displacement vector, $p_i$ ($i = 1, 2, 3$) are the pressures in the fluid phase, $\lambda$ and $\mu$ are the Lamé parameters, $\alpha$, $\beta$, $\mu$ are the constants characterizing the microstructure of the considered elastic medium, $\beta_i$ ($i = 1, 2, 3$) are the effective stress parameters. In the stationary case, the values $p = (p_1, p_2, p_3)^T$ satisfy the following equation

$$\Delta p - Ap = 0, \quad A = \begin{pmatrix} b_1/a_1 & -a_{12}/a_1 & -a_{13}/a_1 \\ -a_{21}/a_2 & b_2/a_2 & -a_{23}/a_2 \\ -a_{31}/a_3 & -a_{32}/a_3 & b_3/a_3 \end{pmatrix}$$

where $a_i = \frac{k_i}{\mu'}$ (for the fluid phase, each phase $i$ carries its respectively permeability $k_i$, $\mu'$ is fluid viscosity), $a_{ij}$ is the fluid transfer rate between phase $i$ and phase $j$, $\Delta$ is the 2D Laplace operator, $b_1 = a_{12} + a_{13}$, $b_2 = a_{21} + a_{23}$, $b_3 = a_{31} + a_{32}$. 

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On the plane \( x_1x_2 \), we introduce the complex variable \( z = x_1 + ix_2 = re^{i\theta} \), \((i^2 = -1)\) and the operators \( \partial_z = 0.5(\partial_1 - i\partial_2) \), \( \partial_z = 0.5(\partial_1 + i\partial_2) \), \( \bar{z} = x_1 - ix_2 \), and \( \Delta = 4\partial_z\partial_{\bar{z}} \).

If relations (2) are substituted into system (1), then system (1) is written in the complex form

\[
2(\mu + \alpha)\partial_z\partial_{\bar{z}}u + (\lambda + \mu - \alpha)\partial_z\partial_{\bar{z}}\theta - 2\alpha i\partial_z\omega - \partial_z(\beta_1p_1 + \beta_2p_2 + \beta_3p_3) = 0, \\
2(\nu + \beta)\partial_z\partial_{\bar{z}}\omega + \alpha(\theta - 2\partial_zu_+) - 2\alpha\omega = 0, \quad (u_+ = u_1 + iu_2). 
\]

\[ (4) \]

2 The general solution of system (3)-(4). In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [7] for system (4).

Equations (3) imply that

\[
p_i = f'(z) + \overline{f'(z)} + l_{11}\chi_1(z, \bar{z}) + l_{22}\chi_2(z, \bar{z}),
\]

where \( f(z) \) is an arbitrary analytic functions of a complex variable \( z \) in the domain \( V \) and \( \chi_\alpha(z, \bar{z}) \) is an arbitrary solution of the Helmholtz equation \( \Delta \chi_\alpha(z, \bar{z}) - \kappa_\alpha\chi_\alpha(z, \bar{z}) = 0 \), \( \kappa_\alpha \) are eigenvalues and \( (l_{11}, l_{21}, l_{31}), (l_{12}, l_{22}, l_{32}) \) are eigenvectors of the matrix \( A \).

Theorem 1. The general solution of the system of equations (4) is represented as follows:

\[
2\mu u_+ = \kappa\varphi(z) - z\varphi'(z) - \psi(z) + \delta^*(f'(z) + \overline{f'(z)}) + \frac{4\mu}{\lambda + 2\mu}\partial_z[\delta_1\chi_1(z, \bar{z}) + \delta_2\chi_2(z, \bar{z})],
\]

\[
2\mu\omega = \frac{2\mu}{\nu + \beta}\chi(z, \bar{z}) - \frac{\kappa + 1}{2}i(\varphi'(z) + \overline{\varphi'(z)}),
\]

where \( \kappa = \frac{\lambda + 3\mu}{\lambda + \mu} \), \( \delta^* = \frac{\mu(\delta_1 + \delta_2 + \delta_3)}{\lambda + 2\mu} \), \( \delta_\alpha := \frac{\mu}{\kappa_\alpha}\beta_1 + \frac{\nu}{\kappa_\alpha}\beta_2 + \frac{\alpha}{\kappa_\alpha}\beta_3 \), \( \varphi(z) \) and \( \psi(z) \) are arbitrary analytic functions of a complex variable \( z \) in the domain \( V \), \( \chi(z, \bar{z}) \) is an arbitrary solution of the Helmholtz equation \( 4\partial_z\partial_{\bar{z}}\chi(z, \bar{z}) - \xi^2\chi(z, \bar{z}) = 0 \), \( \xi^2 := \frac{2\mu}{(\nu + \beta)(\lambda + \alpha)} > 0 \).

3 A problem for a circle. In this section, we solve a concrete boundary value problem for a circle of radius \( R \) (Figure 1). On the boundary of the considered domain the values of pressures \( p_1 \) and \( p_2 \) and the displacement vector are given.

![Figure 1: The circle](image)

We consider the following problem

\[
p_j|_{r=R} = P_j = \sum_{n=-\infty}^{+\infty} A_{nj}e^{in\theta}, \quad A_{nj} = \overline{A}_{-nj}, \quad j = 1, 3, \quad \text{(5)}
\]

\[
2\mu u_+|_{r=R} = 2\mu(G_1 + iG_2) = \sum_{n=-\infty}^{+\infty} B_n e^{in\theta}, \\
2\mu\omega|_{r=R} = G_3 = \sum_{n=-\infty}^{+\infty} C_n e^{in\theta}, \quad C_n = \overline{C}_{-n}. \quad \text{(6)}
\]
Compare the coefficients at identical degrees. We obtain the following systems of equations

\[ a_1 + \bar{a}_1 + l_1 I_0(k_1 R) \alpha_{01} + l_2 I_0(k_2 R) \alpha_{02} = A_{0j}, \quad j = 1, 2, 3, \]

\[ n R^{n-1} a_n + l_1 I_{n-1}(k_1 R) \alpha_{n-11} + l_2 I_{n-1}(k_2 R) \alpha_{n-12} = A_{n-1j}, \quad n > 1. \]

From (7) we can find \( a_1 + \bar{a}_1, \ a_n, \ \alpha_{n1}, \ \alpha_{n2} \).

Now the analytic functions \( \varphi(z) \), \( \psi(z) \) and the metaharmonic functions \( \chi(z, \bar{z}) \) are represented as the series

\[ \varphi(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \psi(z) = \sum_{n=1}^{\infty} c_n z^n, \quad \chi(z, \bar{z}) = \sum_{n=-\infty}^{\infty} \beta_n I_n(\zeta r) e^{i\phi} \]

and are substituted in the boundary conditions (6) we have

\[ \sum_{n=1}^{\infty} (\kappa b_n + \delta^*) R^n e^{i\phi} - (\bar{b}_1 - \delta^* \bar{a}_1) R e^{i\phi} - \sum_{n=0}^{\infty} (n+2)(\bar{b}_{n+2} - \delta^* \bar{a}_{n+2}) R^{n+2} e^{-i\phi} \]

\[ - \sum_{n=0}^{\infty} R^n \beta_n e^{-i\phi} + \zeta i \sum_{n=-\infty}^{\infty} \beta_{n-1} I_n(\zeta r) e^{i\phi} = \sum_{n=-\infty}^{\infty} A'_n e^{i\phi}, \]

\[ \frac{2\mu}{\nu + \beta} \sum_{n=-\infty}^{\infty} \beta_n I_n(\zeta r) e^{i\phi} - \frac{\kappa + 1}{2} i \sum_{n=0}^{\infty} R^n [b_{n+1} e^{i\phi} - \bar{b}_{n+1} e^{-i\phi}] = \sum_{n=-\infty}^{\infty} C_n e^{i\phi}, \]

where \( A'_n = B_n - \frac{4 \mu}{\lambda + 2\mu} \left[ \frac{k_1 \delta}{2} I_n(k_1 R) + \frac{k_2 \delta}{2} I_n(k_2 R) \right] \).

Compare the coefficients at identical degrees. We obtain

\[ R(\kappa b_1 - \bar{b}_1) + \zeta i I_1(\zeta r) \beta_0 = A''_1, \quad R(\kappa \bar{b}_1 - b_1) - \zeta i I_1(\zeta r) \beta_0 = \bar{A}'_1, \]

\[ -\frac{\kappa + 1}{2} i R(b_1 - \bar{b}_1) + \frac{2\mu}{\nu + \beta} I_0(\zeta r) \beta_0 = C_0, \quad (A''_1 = A'_1 - \delta^* R(a_1 + \bar{a}_1)), \]

\[ \kappa R^n b_n + i \zeta I_n(\zeta r) \beta_{n-1} = A'_n - \delta^* R^n a_n, \]
\[
\frac{2\mu}{\nu + \beta} I_{n-1}(\zeta R) \beta_{n-1} - \frac{\kappa + 1}{2} inR^{n-1} b_n = C_{n-1}, \quad (8)
\]

\[
(n + 2)R^{n+2} b_{n+2} + \zeta i^{n-1} I_n(\zeta R) + R^n c_n = (n + 2)\delta^* R^{n+2} a_{n+2} - \bar{A}_n', \quad n \geq 0.
\]

From (8) we can find coefficients \( b_n, \ c_n, \ \beta_n \).

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

Similarly the problem can be solved when on the boundary of the considered domain the values of stresses are given.

**REFERENCES**


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