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BASIC BOUNDARY VALUE PROBLEMS FOR CIRCLE WITH DOUBLE POROSITY *

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Abstract. In this paper plane problems of elasticity for a circle with double porosity is considered. The solutions are represented by means of three analytic functions of a complex variable and one solution of the Helmholtz equation. The problems are solved when the components of the displacement vector is known on the boundary.

Keywords and phrases: Double porosity, the stress tensor, a circle.

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1 Introduction. A theory of consolidation for elastic materials with double porosity was presented in [1-3]. Various issues related to the elastic equilibrium of bodies with double porosities are treated in [4-6].

In this paper we consider the case of a plane deformed state and write the corresponding two-dimensional system of equilibrium equations in the complex form. We construct the general solution of the above-mentioned system of equations by means of three analytic functions of a complex variable and solutions of Helmholtz equations. The obtained analogues of the Kolosov-Muskhelishvili formulas make it possible to solve analytically plane boundary value problems of the elastic equilibrium with double porosity. We solve a boundary value problem for a circle.

2 The plane deformation. Basic equations. Let V be a bounded domain in the Euclidean two-dimensional space E^2 bounded by the contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$. Let $x = (x_1, x_2)$ be the points of space E^2, $\partial_i = \frac{\partial}{\partial x_i}$. Let us assume that the domain V is filled with an isotropic double porosity material.

The system of homogeneous equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows

$$\partial_\alpha \sigma_{\alpha\beta} = 0, \quad (\alpha, \beta = 1, 2) \tag{1}$$

$$\begin{aligned} &\sigma_{11} = \lambda\theta + 2\mu\partial_1 u_1 - \beta_1 p_1 - \beta_2 p_2, \quad \sigma_{22} = \lambda\theta + 2\mu\partial_2 u_2 - \beta_1 p_1 - \beta_2 p_2, \\ &\sigma_{12} = \sigma_{12} = \mu(\partial_1 u_2 + \partial_2 u_1), \quad \theta := \partial_1 u_1 + \partial_2 u_2, \end{aligned} \tag{2}$$

where $\sigma_{\alpha\beta}$ are stress tensor components, p_α $(\alpha = 1, 2)$ are the pressures in the fluid phase, λ and μ are the Lamé parameters, β_α $(\alpha = 1, 2)$ are the effective stress parameters.

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In the stationary case, the values p_1 and p_2 satisfy the following equation

$$\begin{cases} (k_1\Delta-\gamma)p_1+(k_{12}\Delta+\gamma)p_2=0, \\ (k_{21}\Delta-\gamma)p_1+(k_2\Delta+\gamma)p_2=0, \end{cases} \quad in\ V, \tag{3}$$

where γ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, $k_\alpha=\dfrac{\kappa_\alpha}{\mu'}$, $k_{12}=\dfrac{\kappa_{12}}{\mu'}$, $k_{21}=\dfrac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the 2D Laplace operator.

On the plane x_1x_2, we introduce the complex variable $z=x_1+ix_2=re^{i\vartheta}$, $(i^2=-1)$ and the operators $\partial_z=0.5(\partial_1-i\partial_2)$, $\partial_{\bar z}=0.5(\partial_1+i\partial_2)$, $\bar z=x_1-ix_2$, and $\Delta=4\partial_z\partial_{\bar z}$.

If relations (2) are substituted into system (1), then system (1) is written in the complex form

$$2\mu\partial_{\bar z}\partial_z u_+ +(\lambda+\mu)\partial_{\bar z}\theta-\partial_{\bar z}(\beta_1p_1+\beta_2p_2)=0, \quad in\ V, \quad (u_+=u_1+iu_2). \tag{4}$$

3 The general solution of system (3)-(4). In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [7] for system (4).

From system (3) we easily obtain the expressions for the pressures p_1 and p_2

$$p_1=f'(z)+\overline{f'(z)}+(k_2+k_{12})\eta(z,\bar z), \quad p_2=f'(z)+\overline{f'(z)}-(k_1+k_{21})\eta(z,\bar z),$$

where $f(z)$ is an arbitrary analytic functions of a complex variable z in the domain V and $\eta(z,\bar z)$ is an arbitrary solution of the Helmholtz equation

$$4\partial_z\partial_{\bar z}\eta-\zeta^2\eta=0, \quad \zeta^2=\frac{\gamma(k_1+k_2+k_{12}+k_{21})}{k_1k_2-k_{12}k_{21}}.$$

Theorem 1. *The general solution of the system of equations (4) is represented as follows:*

$$2\mu u_+=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}+\frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(f'(z)+\overline{f'(z)})+\delta\partial_{\bar z}\eta(z,\bar z),$$

where

$$\kappa=\frac{\lambda+3\mu}{\lambda+\mu} \quad \delta=\frac{4\mu((k_2+k_{12})\beta_1-(k_1+k_{21})\beta_2)}{(\lambda+2\mu)\zeta^2},$$

$\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z in the domain V.

4 A problem for a circle. In this section, we solve a concrete boundary value problem for a circle of radius R (Figure 1). On the boundary of the considered domain the values of pressures p_1 and p_2 and the displacement vector are given.

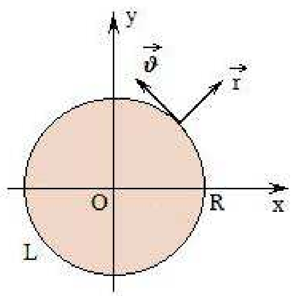

Figure 1: The circle

We consider the following problem

$$p_1 = \sum_{-\infty}^{+\infty} A_n e^{in\vartheta}, \quad |z| = R,$$
$$p_2 = \sum_{-\infty}^{+\infty} B_n e^{in\vartheta}, \quad |z| = R, \tag{5}$$

$$u_+ = \sum_{-\infty}^{+\infty} C_n e^{in\vartheta}, \quad |z| = R. \tag{6}$$

The analytic function $f(z)$ and the metaharmonic function $\eta(z, \bar{z})$ is represented as a series

$$f(z) = \sum_{n=1}^{+\infty} a_n e^{in\vartheta}, \quad \eta(z, \bar{z}) = \sum_{-\infty}^{+\infty} \alpha_n I_n(r\zeta) e^{in\vartheta}, \tag{7}$$

where $I_n(r\zeta)$ is a modified Bessel function of n-th order, and are substituted in the boundary conditions (5) we have

$$\begin{aligned} &\sum_{n=1}^{+\infty} nR^{n-1}\left(a_n e^{i(n-1)\vartheta} + \bar{a}_n e^{-i(n-1)\vartheta}\right) + (k_2 + k_{12}) \sum_{-\infty}^{+\infty} \alpha_n I_n(R\zeta) e^{in\vartheta} = \sum_{-\infty}^{+\infty} A_n e^{in\vartheta}, \\ &\sum_{n=1}^{+\infty} nR^{n-1}\left(a_n e^{i(n-1)\vartheta} + \bar{a}_n e^{-i(n-1)\vartheta}\right) - (k_1 + k_{21}) \sum_{-\infty}^{+\infty} \alpha_n I_n(R\zeta) e^{in\vartheta} = \sum_{-\infty}^{+\infty} B_n e^{in\vartheta}. \end{aligned} \tag{8}$$

Compare the coefficients at identical degrees. We obtain the following system of equations

$$\left\{ \begin{array}{l} a_1 + \bar{a}_1 + (k_2 + k_{12}) I_0 \alpha_0 = A_0, \\ a_1 + \bar{a}_1 - (k_1 + k_{21}) I_0 \alpha_0 = B_0, \end{array} \right. \quad \left\{ \begin{array}{l} nR^{n-1} a_n + (k_2 + k_{12}) I_{n-1} \alpha_{n-1} = A_{n-1}, \\ nR^{n-1} a_n - (k_1 + k_{21}) I_{n-1} \alpha_{n-1} = B_{n-1}. \end{array} \right. \tag{9}$$

The solutions of the system (9) have the following forms:

$$a_n = \frac{(k_1 + k_{21}) A_{n-1} + (k_2 + k_{12}) B_{n-1}}{nR^{n-1}(k_1 + k_2 + k_{12} + k_{21})}, \quad \alpha_n = \frac{A_n - B_n}{(k_1 + k_2 + k_{12} + k_{21}) I_{n-1}}.$$

The analytic functions $\varphi(z)$ and $\psi(z)$ are represented as the series

$$\varphi(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and are substituted in the boundary conditions (6) we have

$$\kappa \sum_{n=1}^{\infty} R^n b_n e^{in\vartheta} - \bar{b}_1 R e^{i\vartheta} - \sum_{n=0}^{\infty} (n+2) R^{n+2} \bar{b}_{n+2} e^{-in\vartheta} - \sum_{n=0}^{\infty} R^n \bar{c}_n e^{-in\vartheta}$$

$$+ \frac{\mu(\beta_1 + \beta_2) R^n}{\lambda + 2\mu} \left[\sum_{n=1}^{\infty} a_n e^{in\vartheta} + \sum_{n=1}^{\infty} \bar{a}_n e^{-i(n-2)\vartheta} \right] - \frac{\delta\zeta}{2} \sum_{-\infty}^{+\infty} \alpha_n I_{n+1} e^{i(n+1)\vartheta} = \sum_{-\infty}^{+\infty} C_n e^{in\vartheta}.$$

Compare the coefficients at identical degrees. We obtain

$$b_n = \frac{1}{\kappa}\left(\frac{C_n}{R^n} - \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}a_n + -\frac{\delta\zeta}{2R^n}\alpha_{n-1}I_n\right), \quad n > 1,$$

$$c_n = \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(n+2)R^2a_{n+2} - \frac{\delta\zeta}{2R^n}\alpha_{n+1}I_n - (n+2)R^2b_{n+2} - \frac{\overline{C}_{-n}}{R^n}, \quad n \geq 0,$$

$$b_1 = \frac{\kappa C_1 + \overline{C}_1}{(\kappa^2-1)R} - \frac{\mu(\beta_1+\beta_2)}{(\lambda+2\mu)(\kappa-1)} + \frac{\delta\zeta}{2R(\kappa-1)}\alpha_0 I_1.$$

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

Similarly the problem can be solved when on the boundary of the considered domain the values of stresses are given.

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