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ON THE UNCONDITIONAL CONVERGENCE OF GENERAL FOURIER SERIES

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Abstract. The paper deals with the problems of unconditional convergence almost everywhere (a.e.) of Fourier series with respect to general orthonormal systems (ONS). The conditions are found to be satisfied by the functions of the orthonormal system so that the Fourier series of every function of finite variation unconditionally converges a.e. The obtained results are best possible in a certain sense.

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1 Introduction.

Banach [1] proved that if the function $f \in L_2([0.1])$ $f(x) \nsim 0$, there exists ONS $(\varphi_n(x))$, such that

$$\lim_{n\to\infty} \sup |S_n(f,x)|_{=+\infty} \; on \; [0.1],$$

where

$$S_n(f,x) = \sum_{k=1}^{n} c_n(f)\varphi_n(x)$$

and

$$c_k(f) = \int_0^1 f(x)\varphi_k(x)dx$$

are Fourier coefficients.

On the other hand, classical systems, such as the trigonometric system, Walsh, Haar, Legendre, Franklin and other systems have the property that the Fourier series of functions with "good" properties have good behaviour.

The problem arises: to select from ONS those systems which have the properties of classical ONS.

Consider the following class of functions: A_1 is the class of continuous functions, A_2 is the class of functions of bounded variation $(V(0,1)), A_3$ is Lip $\alpha,\ 0 < \alpha \le 1,\ A_4$ is the class of absolutely continuous functions, etc. Consider also the following properties of Fourier series: B_1 is convergence almost everywhere, B_2 is a.e. summability by Cesaro methods $(C, \alpha),\ \alpha > 0$, B_3 is unconditional convergence a.e., B_4 is absolute convergence of Fourier coefficients, etc.

The problem arises: what are the conditions to be satisfied by ONS $(\varphi_n(x))$ so that the Fourier series of every function from the class A_i has the property B_j.

In the present paper the conditions are found to be satisfied by ONS $(\varphi_n(x))$ so that the Fourier series of any function from $V(0,1)$ is unconditionally convergent by the system $(\varphi_n(x))$.

The problems of convergence of different types of Fourier series with respect to general ONS have been considered in many works. Note, e.g. [2]-[7]; see also [8]-[12] which are close to the problems, considered in the paper.

Definition 1. Unconditional convergence a.e. on [0,1] of the series $\sum\limits_{n=1}^{\infty} f_n(x)$ means that this series converges a.e. on [0,1] whatever the rearrangement of its members. We have

Theorem A (Menshov [2,p. 350]). If for some $\varepsilon \in (0,1)$

$$\sum_{n=1}^{\infty} |a_n|^{2-\varepsilon} < \infty, \tag{1}$$

then the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

unconditionally converges a.e. on [0.1], where $(\varphi_n(x))$ is any ONS on [0,1].

Let $\varepsilon \in (0,1)$ and let $a = (a_n)$ be some sequence of numbers. Assume

$$D_n(a,\varepsilon) = \left(\sum_{n=1}^{n} |a_n|^{2-\varepsilon} \right)^{\frac{1-\varepsilon}{2-\varepsilon}}$$

and

$$B_n(a,\varepsilon,t) = \max_{1 \le i \le n} | \int_0^{\frac{i}{n}} P_n|(a,\varepsilon,t,x)dx \tag{2}$$

where

$$P_n(a,\varepsilon,t,x) = \sum_{k=1}^{n} |a_k|^{1-\varepsilon} r_k(t) \varphi_k(x),$$

$r_k(t)$ is the Rademacher function (see [2]), p.22) and $\varphi_n(x)$ is ONS on [0,1].

Theorem 1. *Let ONS $(\varphi_n(x))$ on $[0,1]$ satisfy the following conditions:*

a) $\int\limits_0^1 \varphi_n(x)dx = 0$ *beginning with some natural n;*

b) for some $\varepsilon \in (0,1)$ any $t \in [0,1]$ and any sequence $a = (a_n) \in \ell_2$

$$B_n(a,\varepsilon,t) = 0(1) D_n(a,\varepsilon)$$

Then, the Fourier coefficients of any function $f \in v(0,1)$ satisfy condition (1).

From Theorems A and 1 there follows.

Theorem 2. *Let ONS on* $[0,1]$ *satisfy condition: a)of Theorem 1. if for some* $\varepsilon \in (0,1)$ *for any* $t \in [0,1]$ *and for any sequence* $a = (a_n) \in \ell_2$ *condition (3) is satisfied, then the Fourier series of all functions from* $V(0,1)$ *is uconditionally convergent a.e. on* $[0,1]$.

The following theorem shows that Theorem 1 is best possible in a certain sense.

Theorem 3. *Let ONS* $(\varphi_n(x))$ *on* $[0,1]$ *satisfy condition a) of Theorem 1. If for all* $\varepsilon \in (0,1)$, *for some* $t_0 \in [0,1]$ *or for some sequence* $d = (d_n) \in \ell_2$ *the condition*

$$\lim_{n\to\infty} \frac{B_n(d,\varepsilon,t_0)}{D_n(d,\varepsilon)} = +\infty$$

holds, then there exists the absolutely continuous function $f_0(x)$ *such that for all* $\varepsilon \in (0,1)$

$$\sum_{n=1}^{\infty} |c_n(f_0)|^{2-\varepsilon} = +\infty,$$

where

$$c_n(f) = \int_0^1 f_0(x)\varphi_n(x)dx$$

Theorem 4. *From any ONS* $(f_n(x))$ *satisfying condition a) of Theorem 1, one can select the subsystem* $\varphi_{nk}(x) = g_k(x)$ *such that for any function* f *from* $v(0,1)$ *the following condition*

$$\sum_{n=1}^{\infty} |c_n(f)|^{2-\varepsilon} < +\infty$$

holds for some $\varepsilon \in (0,1)$, *where* $c_n(f) = \int_0^1 f(x)\varphi_n(x)dx$

Remark 1. From Theorems A and 4 it follows that any ONS has the subsystem, with respect to which the Fourier series of any function from $V(0,1)$ is unconditionally convergent a.e.

Theorem 5. *Let* $\varphi_n(x))$ *be ONS and let condition a) of Theorem 1 be fulfilled. If*

$$\max_{1\le i\le n} |\int_0^{\frac{i}{n}} \varphi_n(x)dx| = 0\left(\frac{1}{n}\right)$$

then let for ewery $\varepsilon \in (0,1)$, *for any* $t \in [0,1]$ *and any sequence* $a = (a_n) \in \ell_2$ *condition (3) be fulfilled.*

Condition (3) may be easily verified for the trigonometric, walsh and Haer systems (see [2,p 117, 150,69]).

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