Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

ON THE NUMERICAL SOLUTION OF ONE NONLINEAR INTEGRO-DIFFERENTIAL SYSTEM WITH SOURCE TERMS

Mikheil Gagoshidze

Abstract. One nonlinear integro-differential system with source terms is considered. The model arises on mathematical simulation of the process of penetration of a magnetic field into a substance. Initial-boundary value problem with mixed boundary condition is investigated. Finite difference scheme is constructed and studied. Graphical illustrations of numerical experiments are given.

Keywords and phrases: Nonlinear integro-differential system, finite difference scheme.

AMS subject classification (2010): 65N06, 45K05, 35K55.

The purpose of this note is to study the finite difference scheme for one diffusion system of nonlinear partial integro-differential equations. The mentioned system is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell equations [8], describing process of penetration of an electromagnetic field into a substance, to the system of nonlinear integro-differential equations. At first such reduction to the integro-differential model was made in [3]. Later a number of scientists studied proposed in the work above integro-differential models for different cases of magnetic field and different kind of nonlinearity (see, for example, [1] - [10], [13] and references therein).

Let us consider the following initial-boundary value problem with mixed boundary conditions:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[\left(1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)^p\frac{\partial U}{\partial x}\right]+|U|^{q-2}U=0,$$

$$\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left[\left(1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)^p\frac{\partial V}{\partial x}\right]+|V|^{q-2}V=0, \tag{1}$$

$$U(0,t)=V(0,t)=\left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1}=\left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1}=0,$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),$$

where $0<p\leq 1$, $q\geq 2$, U_0 and V_0 are given functions.

System (1) is obtained by adding the source terms to the one-dimensional analog of averaged model which is proposed by Laptev [11]. The basis of system (1) takes place from [1], where reduction of Maxwell [8] system to the integro-differential equation was made. There are many authors who investigate problem (1) and models like this system (see, for example, [1] - [7], [9] - [11], [13]).

In the finite rectangle $(0,1)\times(0,T)$, where T is a positive constant let us study the difference scheme for the initial-boundary value problem (1).

On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$; where $i=0,1,...,M$; $j=0,1,...,N$ with $h=1/M$, $\tau=1/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by (u_i^j,v_i^j) and the exact solution to the problem (1) by (U_i^j,V_i^j). We will use the following known notations [12]:

$$r_{x,i}^j=\frac{r_{i+1}^j-r_i^j}{h},\quad r_{\bar{x},i}^j=\frac{r_i^j-r_{i-1}^j}{h}.$$

Introduce the inner products and norms:

$$(r^j,g^j)=h\sum_{i=1}^{M-1}r_i^jg_i^j,\quad (r^j,g^j]=h\sum_{i=1}^{M}r_i^jg_i^j,$$

$$||r^j\|=(r^j,r^j)^{1/2},\quad ||r^j]|=(r^j,r^j]^{1/2}.$$

For problem (1) let us consider the following finite difference scheme:

$$\frac{u_{j+1}^i-u_j^i}{\tau}-\left\{\left(1+h\tau\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[\left(u_{\bar{x},\ell}^k\right)^2+\left(v_{\bar{x},\ell}^k\right)^2\right]\right)^p u_{\bar{x},i}^{j+1}\right\}_x+\left|u_i^j\right|^{q-2}u_i^j=0,$$

$$\frac{v_{j+1}^i-v_j^i}{\tau}-\left\{\left(1+h\tau\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[\left(u_{\bar{x},\ell}^k\right)^2+\left(v_{\bar{x},\ell}^k\right)^2\right]\right)^p u_{\bar{x},i}^{j+1}\right\}_x+\left|v_i^j\right|^{q-2}v_i^j=0,\qquad(2)$$

$$u_0^j=v_0^j=u_{\bar{x},M}^j=v_{\bar{x},M}^j=0,\quad j=0,1...,N,$$

$$u_i^0=U_{0,i},\quad v_i^0=V_{0,i},\quad i=0,1,...,M.$$

It is not difficult to get the inequalities:

$$\|u^n\|^2+\sum_{j=1}^{n}||u_{\bar{x}}^j]|^2\tau<C,\quad \|v^n\|^2+\sum_{j=1}^{n}||v_{\bar{x}}^j]|^2\tau<C,\qquad(3)$$

where here and below C is a positive constant independent from τ and h.

The a priori estimates (3) guarantee the stability of the scheme (2).

The main result of this note is the following statement.

Theorem 1. *If problem (1) has a sufficiently smooth solution* $(U(x,t),V(x,t))$, *then the solution* $\left(u^j=(u_1^j,u_2^j,...,u_M^j),\ v^j=(v_1^j,v_2^j,...,v_M^j)\right)$, $j=1,2,...,N$, *of the difference scheme (2) tends to solution* $\left(U^j=(U_1^j,U_2^j,...,U_M^j),\ V^j=(V_1^j,V_2^j,...,V_M^j)\right)$, $j=1,2,...,N$ *of continuous problem (1) as* $\tau\to 0$, $h\to 0$ *and the following estimates are true*

$$\|u^j-U^j\|\le C(\tau+h),\quad \|v^j-V^j\|\le C(\tau+h).$$

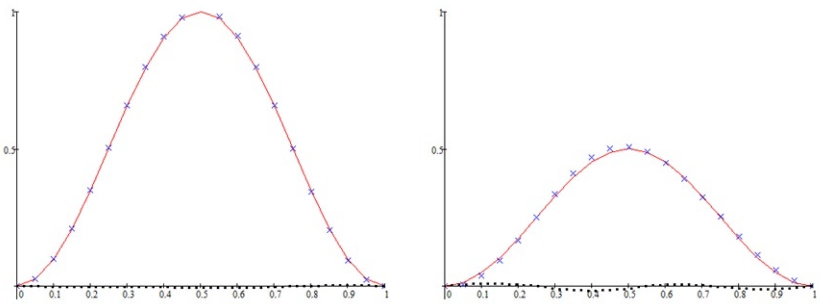

Figure 1: Exact and numerical solution for U (left) and V (right)

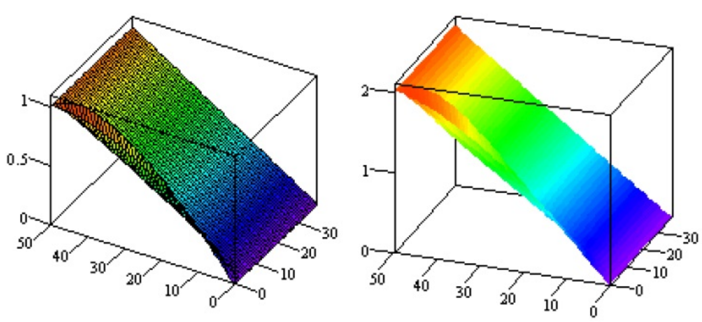

Figure 2: Stabilization of solution

R E F E R E N C E S

1. Aptsiauri, M., Jangveladze T., Kiguradze, Z. Asymptotic behavior of the solution of a system of nonlinear integro-differential equations. *Differ. Uravn.*, **48** (2012), 70-78 (Russian). English translation: *Diff. Equ.*, **48** (2012), 72-80.

2. Dzhangveladze, T. First boundary value problem for a nonlinear equation of parabolic type. *Dokl. Akad. Nauk SSSR*, **269** (1983), 839-842 (Russian). English translation: Soviet Phys. Dokl., **28** (1983), 323-324.

3. Gordeziani, D., Dzhangveladze T,. Korshia, T. Existence and uniqueness of a solution of certain nonlinear parabolic problems. *Differ. Uravn.*, **19** (1983), 1197-1207 (Russian). English translation: Differ. Equ., **19** (1983), 887-895.

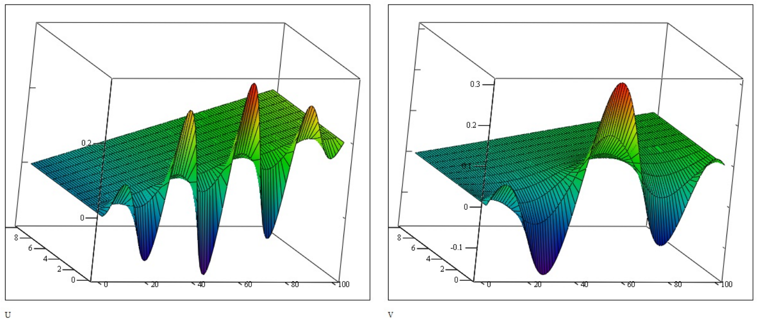

Figure 3: Numerical solutions u and v

4. Jangveladze, T. Convergence of a difference scheme for a nonlinear integro-differential equation. *Proc. I. Vekua Inst. Appl. Math.*, **48** (1998), 38-43.

5. Jangveladze, T. On one class of nonlinear integro-differential equations, *Sem. I.Vekua Inst. Appl. Math., REPORTS*, **23** (1997), 51-87.

6. Jangveladze, T., Kiguradze. Asymptotics for large time of solutions to nonlinear system associated with the penetration of a magnetic field into a substance. *Appl. Math.*, **55** (2010), 441-463.

7. Jangveladze, T., Kiguradze, Z., Neta, B. Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations. *Amsterdam: Elsevier/Academic Press.*, 2016.

8. Landau, L., Lifschitz, E. Electrodynamics of Continuous Media, Course of Theoretical Physics. *Moscow*, 1957.

9. Kratsashvili, M. On finite difference scheme for one integro-differential model with source terms based on Maxwell system, *Bulletin of TICMI*, **21** (2017), 21-32.

10. Jangveladze, T., Kratsashvili, M. Some properties of solution and finite difference scheme for one nonlinear partial differential model based on Maxwell system. *Mem. Differential Equations Math. Phys.,* **42** (2018), 35-48 (accepted).

11. Laptev, G. Quasilinear Evolution Partial Differential Equations with Operator Coefficients (Russian). *Doctoral Dissertation, M. V. Keldish Inst. Appl. Math. Russian Acad. Sci., Moscow*, 1990.

12. Samarskii, A. The Theory of Difference Schemes (Russian). *Nauka, Moscow,* 1977.

13. Sharma, N., Khebchareon, M., Sharma, K., Pani, A. Finite element Galerkin approximations to a class of nonlinear and nonlocal parabolic problems. *Numer. Meth. Part. Diff. Eq.*, **32** (2016), 1232-1264.

Received 04.05.2017; revised 29.09.2017; accepted 01.10.2017.

Author(s) address(es):

Mikheil Gagoshidze
BDO Georgia
Tarkhnishvili str. 2, 0179 Tbilisi, Georgia
E-mail: MishaGagoshidze@gmail.com