ON THE NUMERICAL SOLUTION OF ONE NONLINEAR INTEGRO-DIFFERENTIAL SYSTEM WITH SOURCE TERMS

Mikheil Gagoshidze

Abstract. One nonlinear integro-differential system with source terms is considered. The model arises on mathematical simulation of the process of penetration of a magnetic field into a substance. Initial-boundary value problem with mixed boundary condition is investigated. Finite difference scheme is constructed and studied. Graphical illustrations of numerical experiments are given.

Keywords and phrases: Nonlinear integro-differential system, finite difference scheme.

AMS subject classification (2010): 65N06, 45K05, 35K55.

The purpose of this note is to study the finite difference scheme for one diffusion system of nonlinear partial integro-differential equations. The mentioned system is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell equations [8], describing process of penetration of an electromagnetic field into a substance, to the system of nonlinear integro-differential equations. At first such reduction to the integro-differential model was made in [3]. Later a number of scientists studied proposed in the work above integro-differential models for different cases of magnetic field and different kind of nonlinearity (see, for example, [1] - [10], [13] and references therein).

Let us consider the following initial-boundary value problem with mixed boundary conditions:

\[ \frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left[ \left( 1 + \int_0^1 \int_0^1 \left[ \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial x} \right)^2 \right] dx d\tau \right)^p \frac{\partial U}{\partial x} \right] + |U|^{q-2} U = 0, \]

\[ \frac{\partial V}{\partial t} - \frac{\partial}{\partial x} \left[ \left( 1 + \int_0^1 \int_0^1 \left[ \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial x} \right)^2 \right] dx d\tau \right)^p \frac{\partial V}{\partial x} \right] + |V|^{q-2} V = 0, \]

\[ U(0, t) = V(0, t) = \frac{\partial U(x, t)}{\partial x} \bigg|_{x=1} = \frac{\partial V(x, t)}{\partial x} \bigg|_{x=1} = 0, \]

\[ U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x), \]

where \( 0 < p \leq 1, \quad q \geq 2, \quad U_0 \) and \( V_0 \) are given functions.
System (1) is obtained by adding the source terms to the one-dimensional analog of averaged model which is proposed by Laptev [11]. The basis of system (1) takes place from [1], where reduction of Maxwell [8] system to the integro-differential equation was made. There are many authors who investigate problem (1) and models like this system (see, for example, [1] - [7], [9] - [11], [13]).

In the finite rectangle $(0, 1) \times (0, T)$, where $T$ is a positive constant let us study the difference scheme for the initial-boundary value problem (1).

On $[0, 1] \times [0, T]$ let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$; where $i = 0, 1, ..., M; j = 0, 1, ..., N$ with $h = 1/M, \tau = 1/N$. The initial line is denoted by $j = 0$. The discrete approximation at $(x_i, t_j)$ is designed by $(u^j_{i,j}, v^j_{i,j})$ and the exact solution to the problem (1) by $(U^j_{i,j}, V^j_{i,j})$. We will use the following known notations [12]:

\[
\begin{align*}
    r^j_{x,i} &= r^j_{i+1} - r^j_{i}, \\
    r^j_{x,i} &= r^j_{i} - r^j_{i-1},
\end{align*}
\]

Introduce the inner products and norms:

\[
\begin{align*}
    (r^j, g^j) &= h \sum_{i=1}^{M-1} r^j_i g^j_i, \\
    (r^j, g^j) &= h \sum_{i=1}^{M} r^j_i g^j_i, \\
    ||r^j|| &= (r^j, r^j)^{1/2}, \\
    ||r^j|| &= (r^j, r^j)^{1/2}.
\end{align*}
\]

For problem (1) let us consider the following finite difference scheme:

\[
\begin{align*}
    \frac{u^j_{i+1} - u^j_i}{\tau} &= \left\{ \left( 1 + h\tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} \left[ \left( u^k_{x,\ell} \right)^2 + \left( v^k_{x,\ell} \right)^2 \right] \right)^p u^j_{x,i} \right\} - |u^j_i|^{q-2} u^j_i = 0, \nonumber \\
    \frac{v^j_{i+1} - v^j_i}{\tau} &= \left\{ \left( 1 + h\tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{M} \left[ \left( u^k_{x,\ell} \right)^2 + \left( v^k_{x,\ell} \right)^2 \right] \right)^p u^j_{x,i} \right\} + |v^j_i|^{q-2} v^j_i = 0, \tag{2}
\end{align*}
\]

\[
\begin{align*}
    u^j_0 &= u^j_{x,M} = v^j_{x,M} = 0, & j = 0, 1, ..., N, \\
    u^0_i &= U_{0,i}, & v^0_i = V_{0,i}, & i = 0, 1, ..., M.
\end{align*}
\]

It is not difficult to get the inequalities:

\[
\begin{align*}
    ||u^j||^2 + \sum_{j=1}^{n} ||u^j||^2 \tau < C, \\
    ||v^j||^2 + \sum_{j=1}^{n} ||v^j||^2 \tau < C,
\end{align*}
\]

where here and below $C$ is a positive constant independent from $\tau$ and $h$. The a priori estimates (3) guarantee the stability of the scheme (2).

The main result of this note is the following statement.
**Theorem 1.** If problem (1) has a sufficiently smooth solution \((U(x,t), V(x,t))\), then the solution \(u^j = (u^j_1, u^j_2, ..., u^j_M), \; v^j = (v^j_1, v^j_2, ..., v^j_M)\), \(j = 1, 2, ..., N\), of the difference scheme (2) tends to solution \((U^j = (U^j_1, U^j_2, ..., U^j_M), \; V^j = (V^j_1, V^j_2, ..., V^j_M))\), \(j = 1, 2, ..., N\) of continuous problem (1) as \(\tau \to 0, \; h \to 0\) and the following estimates are true
\[
\|u^j - U^j\| \leq C(\tau + h), \quad \|v^j - V^j\| \leq C(\tau + h).
\]

![Figure 1: Exact and numerical solution for U (left) and V (right)](image1.png)

![Figure 2: Stabilization of solution](image2.png)

**References**


Figure 3: Numerical solutions $u$ and $v$


6. JANGVELADZE, T., KIGURADZE. Asymptotics for large time of solutions to nonlinear system associated with the penetration of a magnetic field into a substance. *Appl. Math.*, **55** (2010), 441-463.


Received 04.05.2017; revised 29.09.2017; accepted 01.10.2017.

Author(s) address(es):

Mikheil Gagoshidze
BDO Georgia
Tarkhnishvili str. 2, 0179 Tbilisi, Georgia
E-mail: MishaGagoshidze@gmail.com