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# ON THE MULTIDIMENSIONAL FINANCIAL (B,S)-MARKET

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**Abstract**. In the paper one particular model of the discrete financial market with $k$ bonds and one stock is considered. The interest rate dependent on time and related martingale measure are constructed. The relationship between martingale measure, arbitrage and completeness of financial market is established.

**Keywords and phrases**: Financial (B,S)-market, martingale measure, arbitrage, completeness, European option.

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**1 The financial market model and parameters.** Let us consider the financial market in discrete time with one risky asset $S$ and $k$ number of riskless $B_n^{(1)}, B_n^{(2)}, \dots, B_n^{(k)}$, bonds. The prices of the assets $B^(i)$ and $S$ are given by the following recurrent equalities

$$S_n = (1+\rho_n)S_{n-1}, \quad S_0 > 0, \tag{1}$$

$$B_n^{(1)} = (1+r^{(1)})B_{n-1}^{(1)}, \cdots, B_n^{(k)} = (1+r^{(k)})B_{n-1}^{(k)}, \tag{2}$$

$n = 0, 1, 2, \dots, N$. In (2) interest rates $r^{(i)}, i = 1, \dots, k$ and $B_0^{(1)}, \dots, B_0^{(k)}$ are positive constants. In (1.1), which defines price of the stock $S$, $\rho_n$ is the sequence of independent, identically distributed random variables, that take only two values $a$ and $b$, $-1 < a < b$, with probabilities $p > 0$ and $1 - p$ respectively, [1, 2]. At that $a < r^{(i)} < b, i = 1, \dots, k$.

It is assumed that $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ is a stochastic basis, where $\mathcal{F}_n = \sigma\{S_0, \dots, S_n\}$- is the minimal $\sigma$-algebra generated by $S_0, \dots, S_n$.

Considering a model with several bank accounts is justified by the fact, that banks have different interest rates and money deposit or borrowing for investors is favorable from various banks.

Now we introduce the interest rate $r_n$, which is combination of $r^{(1)}, r^{(2)}, \dots, r^{(k)}$ and dependent on time. Suppose, that $B_n = B_n^{(1)} + \ldots + B_n^{(k)}$ and

$$B_n = (1+r_n)B_{n-1}, \tag{3}$$

$r_n > 0$ and consider the financial market $(B, S) = (B_n, S_n), n = 0, 1, 2, \dots, N$.

Let $\pi_0 = (\beta_n^{(1)}, \dots, \beta_n^{(k)}, \gamma_n)$ be the portfolio of investor, where $\beta_n^{(1)}, \dots, \beta_n^{(k)}$ and $\gamma_n$ are quantities of the assets $B^{(1)}, \dots, B^{(k)}$ and $S$ respectively, at the moment $n$. Then the related capital is

$$X_n^{\pi_0} = \beta_n^{(1)}(1+r^{(1)})B_{n-1}^{(1)} + \ldots + \beta_n^{(k)}(1+r^{(k)})B_{n-1}^{(k)} + \gamma_n S_n. \tag{4}$$

On the other hand assume, that the portfolio $\pi = (\beta_n, \gamma_n)$, where $\beta_n$ is the quantity of the asset $B$ at the moment $n$, gives the capital $X_n^{\pi} = X_n^{\pi_0}$. Therefore

$$X_n^{\pi} = \beta_n B_n + \gamma_n S_n = \beta_n(1 + r_n)(B_{n-1}^{(1)} + \ldots + B_{n-1}^{(k)}) + \gamma_n S_n, \tag{5}$$

if in addition we suppose $\beta_n^{(1)} = \ldots = \beta_n^{(k)} = \beta_n$, then from (4) and (5) we obtain

$$r_n = \frac{r^{(1)} B_{n-1}^{(1)} + \cdots + r^{(k)} B_{n-1}^{(k)}}{B_{n-1}^{(1)} + \cdots + B_{n-1}^{(k)}}. \tag{6}$$

**2 Main results and theorems.** The following theorem defines martingale criterion for the measure $P^* \in \mathbb{P}^*$.

**Theorem 1.** *In the model (1)-(3) of financial $(B, S)$-market, with respect to probability measure $P^*$ we have the following equivalence*

$$R_n = \frac{S_n}{B_n} \quad \textit{is a martingale} \Leftrightarrow \sum_{k=0}^{n} (\rho_k - r_k) - \textit{is a martingale},$$

*where $r_k$ is defined by relation (6).*

*Proof.* We introduce the following notations

$$U_n = \sum_{k=0}^{n} r_k, \quad V_n = \sum_{k=0}^{n} \rho_k.$$

With these values prices of bonds $B_n$ and stocks $S_n$ can be written in the form of stochastic exponents

$$B_n = B_0 \mathcal{E}(U), \quad S_n = S_0 \mathcal{E}(V),$$

where stochastic exponents

$$\mathcal{E}_n(U) = \prod_{k=1}^{n} (1 + \Delta U_k), \quad \mathcal{E}_0(U) = 1,$$

$$\mathcal{E}_n(V) = \prod_{k=1}^{n} (1 + \Delta V_k), \quad \mathcal{E}_0(V) = 1.$$

Further, according to the stochastic exponents properties and by [Theorem 2.5, 3], we can write

$$R_n = \frac{S_n}{B_n} = R_0 \mathcal{E}_n(V) \mathcal{E}_n^{-1}(U) = R_0 \mathcal{E}_n\left( \sum_{k=1}^{n} \frac{\Delta V_k - \Delta U_k}{1 + \Delta U_k} \right).$$

From the last equality it follows that $R_n$ is a local martingale if and only if the sequence $\sum_{k=0}^{n} (\rho_k - r_k)$ is a local martingale. $\square$

Of course, it is of interest to find out the relationship between arbitration of $(B, S)$-market and the martingale property of probability measure $P^* \in \mathbb{P}^*$. The following theorem gives an answer to the question.

**Theorem 2.** *Suppose that in the model (1)-(3) of financial $(B, S)$-market, deterministic sequence $r = (r_n)$ is such, that $r_n > -1, n \in \mathbb{N}$. Then*

$$\mathbb{P}^* \neq \emptyset \Leftrightarrow SF_{arb} = \emptyset.$$

*Proof.* Implication $(\Rightarrow)$. Let $P^* \in \mathbb{P}^*$. Then for any self-financing strategy $\pi \in SF$, we have

$$\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n = r_n X_{n-1}^\pi + \gamma_n S_{n-1}(\rho_n - r_n).$$

Hence, since of $U_n$ is deterministic, it follows from the martingale property of $P^*$ and Theorem 1, that if $X_0^\pi = 0$, then

$$E^* X_n^\pi = \mathcal{E}_n(U) E^* = X_0^\pi = 0, \quad n \in \mathbb{N}. \tag{7}$$

Suppose the opposite, that $SF_{arb} \neq \emptyset$ and $\pi \in SF_{arb}$. Then, since the measure $P$ and $P^*$ are equivalent, we obtain $E^* X_n^\pi > 0$, which contradicts (7). Implication $\Rightarrow$ proved.

Implication $(\Leftarrow)$. Let $SF_{arb} = \emptyset$. Note, that the proof of this fact, as in [3], is reduced to the proof of the following equation

$$E^* \left( \frac{S_\tau}{B_\tau} - \frac{S_0}{B_0} \right) = 0, \tag{8}$$

where $\tau = \tau(\omega)$ is the stopping time with values $0, 1, \ldots, N$, and $(S_n/B_n, \mathcal{F}_n, P^*)$ is a martingale. Indeed, we can choose the stopping time $\tau^*$ and construct the sequence $\pi^* = (\pi_n^*)$, that $E^* X_N^\pi = 0$ [3]. Then it is easy to see, that

$$0 = E^* X_N^\pi = E^*(\beta_N^* B_N + \gamma_N^* S_N) = B_N E^* \left( \frac{S_{\tau^*}}{B_{\tau^*}} - \frac{S_0}{B_0} \right).$$

Since $B_N \neq 0$, then the last equality proves (8), implication $(\Leftarrow)$ and consequently Theorem 2. □

In the theorem below there is connection between existence of unique martingale probability measure and completeness of financial market.

**Theorem 3.** *Let the class of martingale measures be nonempty $\mathbb{P}^* \neq \emptyset$ and $P^* \in \mathbb{P}^*$. Then the following statements are equivalent:*

*1) $(B, S)$-market is complete;*

*2) The measure $P^*$ is unique element of $\mathbb{P}^*$.*

*Proof.* 1)$\Rightarrow$ 2). Assume, that there are two measures $P^* \in \mathbb{P}^*$, $P^{**} \in \mathbb{P}^*$ and $P^*(A) \neq P^{**}(A)$, $A \in \mathcal{F}$. Let $f(\omega) = I_A(\omega)B_N$, then there is $\pi \in SF$ such, that $P\{X_N^\pi = I_A B_N\} = 1$ and

$$P^*\{X_N^\pi = I_A B_N\} = P^{**}\{X_N^\pi = I_A B_N\} = 1,$$

$$E^* \frac{X_N^\pi}{B_N} = \frac{X_0^\pi}{B_0}, \quad E^{**} \frac{X_N^\pi}{B_N} = \frac{X_0^\pi}{B_0}.$$

Therefore $E^* I_A = E^{**} I_A$ and $P^*(A) = P^{**}(A)$, so we obtain contradiction. Implication 1)$\Rightarrow$ 2) is proved.

2)$\Rightarrow$ 1). We have to show, that if the measure $P^* \in \mathbb{P}^*$ is a unique martingale measure, then $(B, S)$-market is complete. The proof of this implication is analogous to the proof of [Theorem 3.3, 3]. In particular, on the space $(\Omega, \mathcal{F})$ two sets of random variables are defined

$$\Phi_1 = \{\xi : \text{there is } \pi \in SF \text{ that } X_0^\pi = 0, X_N^\pi = \xi\},$$

$$\Phi_2 = \{\xi : E^*\xi = 0\},$$

and it can be proved [3], that the following implications are fulfilled

$$2) \Rightarrow \Phi_1 = \Phi_2 \Rightarrow 1).$$

So, implication 2)$\Rightarrow$ 1) is valid. $\square$

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