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ON THE SOLVABILITY OF THE FIRST DARBOUX PROBLEM FOR ONE CLASS OF SECOND ORDER NONLINEAR HYPERBOLIC SYSTEMS

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Abstract. The first Darboux problem for one class of nonlinear second order hyperbolic systems is considered.The questions of existence and nonexistence of global solution of this problem are investigated.

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In the plane of variables x and t let us consider the second order hyperbolic system of the form

$$Lu := u_{tt} - u_{xx} + A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t,u) = F(x,t), \qquad (1)$$

where A, B, C are given $n-th$ order matrices, $f = (f_1, ..., f_n)$ and $F = (F_1, ..., F_n)$ are given, and $u = (u_1, ..., u_n)$ is an unknown vector-function, $n \geq 2$.

Let us denote by $D_T := \{(x,t) \in R^2 : 0 < x < t < T\}, T = const > 0$ the angular domain, lying in the characteristic angle $\{(x,t) \in R^2 : t > |x|\}$ and bounded by the characteristic segment

$$\gamma_{1,T} : x = t, 0 \leq t \leq T$$

and noncharacteristic segments

$$\gamma_{2,T} : x = 0, 0 \leq t \leq T$$

and

$$\gamma_{3,T} : t = T, 0 \leq x \leq T.$$

For system (1) in the domain D_T consider the following problem: find in the domain D_T a solution $u = u(x,t)$ of system (1), which satisfies the following boundary conditions:

$$u|_{\gamma_{i,T}} = \varphi_i, \ \ i = 1, 2, \qquad (2)$$

where $\varphi_i, i = 1, 2,$ are given vector - functions, which satisfy the agreement condition

$$\varphi_1(0,0) = \varphi_2(0,0)$$

at the common point O(0,0). With $T = \infty$ we will have $D_\infty : t > |x|, x > 0$ and

$$\gamma_{1,\infty} : x = t, \ 0 \leq t < T,$$

$$\gamma_{2,\infty} : x = 0, \ 0 \leq t \leq T.$$

Definition 1. Let $A, B, C, F \in C(\overline{D}_T), f \in C(\overline{D}_T \times R^n)$ and $\varphi_i \in C^1(\gamma_{i,T}),\ i = 1, 2$. Vector - function u is called a generalized solution of problem (1), (2) of class C in the domain D_T, if $u \in C(\overline{D}_T)$ and there exists a sequence of vector - functions $u^m \in C^2(\overline{D}_T)$ such that $u^m \to u$ as $Lu^m \to F$ in the space $C(\overline{D}_T)$ and $u^m|_{\gamma_{i,T}} \to \varphi_i$ in the space $C^1(\gamma_{i,T}),\ i = 1, 2$, when $m \to \infty$.

Definition 2. Let $A, B, C, F \in C(\overline{D}_\infty), f \in C(\overline{D}_\infty \times R^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty}),\ i = 1, 2$. We say that problem (1), (2) is locally solvable in the class C , if there exists a number $T_0 = T_0(F, \varphi_1, \varphi_2) > 0$ such that for any positive number $T < T_0$ problem (1), (2) has at least one generalized solution of class C in the domain D_T in the sense of Definition 1.

Definition 3. Let $A, B, C, F \in C(\overline{D}_\infty), f \in C(\overline{D}_\infty \times R^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty}),\ i = 1, 2$. We say that problem (1), (2) is globally solvable in the class C, if for any positive T problem (1), (2) has at least one generalized solution of the class C in the domain D_T.

Definition 4. Let $A, B, C, F \in C(\overline{D}_\infty), f \in C(\overline{D}_\infty \times R^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty}),\ i = 1, 2$. Vector - function $u \in C(\overline{D}_\infty)$ is called a global generalized solution of problem (1), (2) of the class C, if for any positive T $u|_{D_T}$ is a generalized solution of problem (1), (2) of the class C in the domain D_T.

Let us note that in the linear case, i.e. when $f = 0$, problem (1), (2) is posed correctly [1 -3]. In the scalar case, when passing to the nonlinear equation, although this problem remains locally solvable in the sense of Definition 2, in the specific cases it is not globally solvable in the sense of Definition 3 [4, 5]. It should also be noted that under the assumptions made above relative to data of problem (1), (2) in the case, when an increase of the nonlinearity of vector - function $f = f(x, t, u)$ with respect to u is not higher than the first order, i.e.:

$$\|f(x, t, u)\| \leq M_1 + M_2 \|u\|,\ \ M_i = const \geq 0,\ i = 1, 2, \tag{3}$$

this problem is globally solvable in the sense of Definition 3. Indeed, in this case for generalized solution of problem (1), (2) of the class C in the domain D_T it the following a priori estimate

$$\|u\|_{C(\overline{D}_T)} \leq\ c_1\|F\|_{C(\overline{D}_T)} +\ c_2\|\varphi_1\|_{C^1(\gamma_{1,T})}\ +\ c_3\|\varphi_2\|_{C^1(\gamma_{2,T})}\ +\ c_4, \tag{4}$$

is valid where nonnegative constants $c_i = c_i(M_0, M_1, M_2, T),\ i = 1, 2, 3, 4,$ do not depend on u, F and φ_1, φ_2, besides,$c_i > 0,\ i = 1, 2, 3,$ and

$$M_0 = \sup_{(x,t)\in D_T} \max_{1\leq i,j\leq n} (\max\{|A_{ij}(x, t)|, |B_{ij}(x, t)|, |C_{ij}(x, t)|\}).$$

Passing to new independent variables ξ and η:

$$\xi\ = {}^1\!/\!_2(t + x), \quad \eta\ = {}^1\!/\!_2(t - x),$$

problem (1), (2) will take the form:

$$L_1 v := v_{\xi\eta} + A_1(\xi,\eta)v_\xi + B_1(\xi,\eta)v_\eta + C_1(\xi,\eta)v + f_1(\xi,\eta,v) = F_1(\xi,\eta),\ (\xi,\eta) \in G_T,$$

$$v|_{OP_1:\ \eta=0,\ 0\le\xi\le T} = \psi_1(\xi),\ 0 \le \xi \le T,$$

$$v\big|_{OP_2:\ \xi=\eta,\ 0\le\eta\le {}^1\!/_2 T} = \psi_2(\eta),\ 0 \le \eta \le {}^1\!/_2 T,$$

in the domain $G_T = OP_1P_2$ of the plane $O = O_{\xi\eta}$, where $P_1 = P_1(T,0),\ P_2 = P_2({}^1\!/_2 T, {}^1\!/_2 T)$ with respect to a new unknown vector-function $v(\xi,\eta) = u(\xi-\eta, \xi+\eta)$. Using the method of characteristics, the obtained problem can be reduced to the following equation

$$v(\xi,\eta) = (L_0 v)(\xi,\eta),$$

where L_0 is a nonlinear integral Volterra type operator acting in the space $(\overline{G}_T)$, besides, this operator is also continuous and compact in the same space, whence, taking into account a priori estimate of u and the equality $v(\xi,\eta) = u(\xi-\eta,\xi+\eta)$, from the Lerray-Schauder's theorem the existence of a generalized solution of the problem (1), (2) of the class C in the domain D_T follows.

Below we consider the cases, when condition (3) is violated, i.e.

$$\overline{\lim_{\|u\|\to\infty}} \frac{\|f(x,t,u)\|}{\|u\|} = \infty$$

and the problem is not globally solvable, in particular it does not have the global generalized solution in the domain D_∞ in the sense of Definition 4.

Theorem 1. *Let* $A = B = C = 0,\ f = f(u) \in C(R^n),\ F \in C(\overline{D_\infty}),\ \varphi_i = 0,\ i = 1,2.$ *There exist numbers* $l_1, ..., l_n,\ \sum_{i=1}^n |l_i| \ne 0$ *such that*

$$\sum_{i=1}^n l_i f_i(u) \ \le\ c_0 \ -\ c_1 \Big|\sum\nolimits_{i=1}^n l_i u_i\Big|^\beta,\ \beta = const > 1, \tag{4}$$

where $c_0,\ c_1 = const,\ c_1 > 0$. *Function* $F_0 = \sum_{i=1}^n l_i F_i - c_0$ *satisfies the following conditions:* $F_0 \ge 0,\quad F(x,t)|_{t\ge1} \ \ge\ c_2 t^{-k};\ c_2 = const >\ 0,\ \ 0 \le k = const \le 2.$ *Then there exists a positive number* $T_0 = T_0(F)$, *such that problem (1), (2) does not have the generalized solution of the class* C *in the domain* D_T, *when* $T > T_0$.

Corollary 1. *From the above said and the theorem it follows that when the conditions of this theorem are fulfilled, though this problem is locally solvable, it has no global generalized solution of the class* C *in the domain* D_∞.

Now we will give one class of nonlinear vector - functions $f = f(u)$ for which condition (4) is satisfied:

$$f_i(u_1, ..., u_n) \ =\ \sum\nolimits_{j=1}^n a_{ij}|u_j|^{\beta_{ij}} \ +\ b_i,\qquad i = 1, ..., n, \tag{5}$$

where $a_{i\,j} = const > \ 0$, $b_i = const$, $\beta_{i\,j} = const > 1$; $i, j = 1, ..., n$. In this case instead of $l_i, i = 1, ..., n$, we can take $l_1 = l_2 = ... = l_n = -1$. Indeed, let us choose $\beta = const$ such that $1 < \beta < \beta_{i\,j}$; $i, j = 1, ..., n$. It is easy to verify that $|s|^{\beta_{ij}} \geq |s|^{\beta} - 1 \ \forall s \in (-\infty, \infty)$. Using the well-known inequality [6]

$$\sum_{i=1}^{n} |y_i|^{\beta} \ \geq \ n^{1-\beta} \left| \sum_{i=1}^{n} y_i \right|^{\beta} \ \forall y = (y_1, ..., y_n) \in R^n, \beta = const > 1,$$

we obtain

$$\sum_{i=1}^{n} f_i(u_1, ..., u_n) \geq \ a_0 \sum_{i,j=1}^{n} |u_j|^{\beta_{i\,j}} + \sum_{i=1}^{n} b_i \ \geq \ a_0 \sum_{i,j=1}^{n} (|u_j|^{\beta} \ - \ 1) + \sum_{i=1}^{n} b_i$$

$$= a_0 n \sum_{j=1}^{n} |u_j|^{\beta} \ - \ a_0 n^2 \ + \ \sum_{i=1}^{n} b_i \ \geq \ \ a_0\, n^{2-\beta} \left| \sum_{j=1}^{n} u_j \right|^{\beta} \ + \ \sum_{i=1}^{n} b_i \ - \ a_0 n^2,$$

$$a_0 \ = \ \min_{i,j} a_{ij} \ > \ 0.$$

From here the inequality (4) with $l_1 = l_2 = ... = l_n = -1$, $c_0 = a_0 n^2 - \sum_{i=1}^{n} b_i$, $c_1 = a_0 n^{2-\beta} > 0$ follows.

Let us note also that the vector - function $f = f(u)$ presented by equalities (5) also satisfies condition (4) where $l_1 = l_2 = ... = l_n = -1$, and coefficients a_{ij} meet less restrictive conditions: $a_{i\,j} = const \geq 0$, but $a_{ik_i} > 0$, where $k_1, ..., k_n$ represents any permutation of numbers $1, 2, ..., n$.

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