Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

ABOUT ONE NON-LOCAL CONTACT PROBLEM FOR ONE DIMENSIONAL HEAT EQUATION *

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Abstract. In the present work, the non-local initial-boundary contact problems for one dimensional parabolic type equation is considered. For the stated problem, the existence and uniqueness of the solution is proved. The iteration process is constructed, which allows one to reduce the solution of the initial non-classical problem to the solution of a sequence of classical Cauchy-Dirichlet problems. The convergence of the proposed iterative process is proved; the speed of convergence is estimated. On the basis of this algorithm the method for numerical solution of the initial problem is described.

Keywords and phrases: Parabolic equation, nonlocal problem, contact problem.

AMS subject classification (2010): 35K20, 34B10, 58J35.

1 Introduction. Nonlocal boundary and initial-boundary contact problems represent very interesting generalization of classical problems and, at the same time, they arise in a natural way during the construction of mathematical models of real processes. Investigation of these problems, construction and analysis of algorithms, among them of parallel algorithms, is an actual direction of applied and computational mathematics (see [1-6] and cited there references).

2 Statement of the problem. Let us consider the domain $D = \{(x,t)| -1 \leq x \leq 1,\ 0 < t \leq T\}$ two half-lines $x = \bar{x}^-$ and $x = \bar{x}^+$

We consider the following problem: find the continuous functions

$$u^-(x,t) \in C^{2,1}\left((-1,0) \times (0,T]\right) \cap C^{1,0}\left([1,0] \times [0,T]\right),$$
$$u^+(x,t) \in C^{2,1}\left((1,0) \times (0,T]\right) \cap C^{1,0}\left([0,1] \times [0,T]\right)$$

which satisfies the equations

$$\frac{\partial u^-}{\partial t} - \frac{\partial^2 u^-}{\partial x^2} = f^-(x,t), \ \text{ if } \ (x,t) \in (-1,0) \times (0,T], \tag{1}$$

$$\frac{\partial u^+}{\partial t} - \frac{\partial^2 u^+}{\partial x^2} = f^+(x,t), \ \text{ if } \ (x,t) \in (0,1) \times (0,T], \tag{2}$$

*This work was supported by the financial support of National Fund for Science of Shota Rustaveli, the project No.: FR/312/4-150/14.

the initial conditions

$$u^{-}(x,0)=u_0^{-}(x),\;\; x\in[-1,0],\;\; u^{+}(x,0)=u_0^{+}(x),\;\; x\in[0,1], \tag{3}$$

the boundary conditions

$$u^{-}(-1,t)=\varphi^{-}(t),\;\; u^{+}(1,t)=\varphi^{+}(t),\;\; 0\le t\le T, \tag{4}$$

and the nonlocal contact condition

$$\begin{aligned} u^{-}(0,t)=u^{+}(0,t)=u_0(t)=\mu^{-}u^{-}\left(\bar{x}^{-},t\right)+\mu^{+}u^{+}\left(\bar{x}^{+},t\right)+\varphi_0(t),\\ 0\le t\le T, \end{aligned} \tag{5}$$

where $\mu^{-}=\text{const}>0$, $\mu^{+}=\text{const}>0$, $f^{-}(\cdot)$ and $f^{+}(\cdot)$, $\varphi^{-}(\cdot)$, $\varphi^{+}(\cdot)$, $u_0^{-}(\cdot)$ and $u_0^{+}(\cdot)$ are known sufficiently smooth functions.

We will call the problem (1)-(5) nonlocal contact problem for heat equation.

3 Uniqueness of a solution of problem (1)-(5).

Theorem 1. *If the regular solution of problem (1)-(5) exists and condition $\mu^{-}+\mu^{+}\le 1$ is fulfilled, then the solution is unique.*

Proof. Suppose that problem (1)-(5) has two regular solutions: $v^{-}(x,t)$, $v^{+}(x,t)$ and $w^{-}(x,t)$, $w^{+}(x,t)$. Then the functions

$$z^{-}(x,t)=v^{-}(x,t)-w^{-}(x,t)\;\text{ and }\; z^{+}(x,t)=v^{+}(x,t)-w^{+}(x,t)$$

are the solution of the following problem:

$$\frac{\partial z^{-}}{\partial t}-\frac{\partial^2 z^{-}}{\partial x^2}=0,\;\text{ if }\;(x,t)\in(-1,0)\times(0,T], \tag{6}$$

$$\frac{\partial z^{+}}{\partial t}-\frac{\partial^2 z^{+}}{\partial x^2}=0,\;\text{ if }\;(x,t)\in(0,1)\times(0,T], \tag{7}$$

$$z^{-}(x,0)=0,\;\; x\in[-1,0],\;\; z^{+}(x,0)=0,\;\; x\in[0,1], \tag{8}$$

$$z^{-}(-1,t)=0,\;\; z^{+}(1,t)=0,\;\; 0\le t\le T, \tag{9}$$

$$\begin{aligned} z^{-}(0,t)=z^{+}(0,t)=z_0(t)=\mu^{-}z^{-}\left(\bar{x}^{-},t\right)+\mu^{+}z^{+}\left(\bar{x}^{+},t\right),\\ 0\le t\le T. \end{aligned} \tag{10}$$

If we suppose $\mu^{-}+\mu^{+}\le 1$, from the equality (10) we will receive:

$$\mathbf{max}\,|z_0(t)|\le \mathbf{max}\left|z^{-}\left(\bar{x}^{-},t\right)\right|\;\text{ or }\;\mathbf{max}\,|z_0(t)|\le \mathbf{max}\left|z^{+}\left(\bar{x}^{+},z\right)\right|,\;\; 0\le t\le T.$$

Taking into account the maximum principle for Heat Equation [8], we can obtain $z^{-}(x,t)\equiv 0$, $(x,t)\in[-1,0]\times[0,T]$, and $z^{+}(x,t)\equiv 0$, $(x,t)\in[0,1]\times[0,T]$. $\square$

4 Existence of a solution of problem (1)-(5). Let us consider the following iteration process for the problem (11)-(16):

$$\left[\frac{\partial u^-}{\partial t}\right]^{(k)} - \left[\frac{\partial^2 u^-}{\partial x^2}\right]^{(k)} = f^-(x,t), \text{ if } (x,t) \in (-1,0) \times (0,T], \tag{11}$$

$$\left[\frac{\partial u^+}{\partial t}\right]^{(k)} - \left[\frac{\partial^2 u^+}{\partial x^2}\right]^{(k)} = f^+(x,t), \text{ if } (x,t) \in (0,1) \times (0,T], \tag{12}$$

$$\begin{aligned} \left[u^-(x,0)\right]^{(k)} &= u_0^-(x), \quad x \in [-1,0], \\ \left[u^+(x,0)\right]^{(k)} &= u_0^+(x), \quad x \in [0,1], \end{aligned} \tag{13}$$

$$\left[u^-(-1,t)\right]^{(k)} = \varphi^-(t), \quad \left[u^+(1,t)\right]^{(k)} = \varphi^+(t), \quad 0 \le t \le T, \tag{14}$$

$$\begin{aligned} \left[u^-(0,t)\right]^{(k)} = \left[u^+(0,t)\right]^{(k)} = \mu^- \left[u^-\left(\bar{x}^-,t\right)\right]^{(k-1)} \\ + \mu^+ \left[u^+\left(\bar{x}^+,t\right)\right]^{(k-1)} + \varphi_0(t), \quad 0 \le t \le T, \end{aligned} \tag{15}$$

where $k = 0, 1, 2, \ldots$ and

$$\left[u^-\left(\bar{x}^-,t\right)\right]^{(-1)} = 0, \quad \left[u^+\left(\bar{x}^+,t\right)\right]^{(-1)} = 0. \tag{16}$$

The following theorem is true.

Theorem 2. *If the regular solution of problem (1)-(5) exists and $\mu^- > 0$, $\mu^+ > 0$, $\mu^- + \mu^+ \le 1$, then the iteration process (11)-(16) converges to this solution at the rate of an infinitely decreasing geometric progression.*

Remark 1. By using the iteration algorithm (11)-(16) one can reduce the solution of a non-classical contact problem (1)-(5) to the solution of a sequence of classical Cauchy-Dirichlet problems. This algorithm is suitable for implementation on computers with parallel processors.

Let us prove the existence of a solution of problem (1)-(5). For this purpose, we consider the iterative process (11)-(16) and introduce the notations

$$\left[\varepsilon^-(x,t)\right]^{(k)} = \left[u^-(x,t)\right]^{(k)} - \left[u^-(x,t)\right]^{(k-1)}, \quad (x,t) \in (-1,0) \times (0,T],$$

$$\left[\varepsilon^+(x,t)\right]^{(k)} = \left[u^+(x,t)\right]^{(k)} - \left[u^+(x,t)\right]^{(k-1)}, \quad (x,t) \in (0,1) \times (0,T].$$

Then we can obtain the following estimation:

$$\max_{0 \le t \le T} \left| \left[\varepsilon^-(0,t)\right]^{(k)} \right| \to 0, \quad \max_{0 \le t \le T} \left| \left[\varepsilon^+(0,t)\right]^{(k)} \right| \to 0, \text{ when } k \to \infty.$$

Thus,

$$\left[u^-(0,t)\right]^{(k)} - \left[u^-(0,t)\right]^{(k-1)} \to 0 \text{ and } \left[u^+(0,t)\right]^{(k)} - \left[u^+(0,t)\right]^{(k-1)} \to 0, \text{ when } k \to \infty$$

This means that the sequences $\left\{[u^-(0,t)]^{(k)}\right\}$ and $\left\{[u^+(0,t)]^{(k)}\right\}$ converge uniformly, $0 \le t \le T$. Using the Harnak's first theorem for Heat Equation [8], one can conclude that the sequences of the functions $\left\{[u^-(x,t)]^{(k)}\right\}$ and $\left\{[u^+(x,t)]^{(k)}\right\}$ converge uniformly to the regular solutions of the problem (1)-(5) - $u^-(0,t)$ and $u^+(0,t)$, respectively, for $(x,t) \in (-1,0) \times (0,T]$ and $(x,t) \in (0,1) \times (0,T]$.

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Received 13.05.2017; revised 21.09.2017; accepted 15.10.2017.

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