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APPLICATION OF FBSDE IN OPTIMAL INVESTMENT PROBLEM

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Abstract. The wealth maximization problem for random utility defined on the half-line is considered. For the solution of this problem the system of Forward Backward Stochastic Differential Equations is derived.

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We consider a financial market model, where the dynamics of asset prices is described by the continuous semimartingale S defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with continuous filtration $F = (F_t, t \in [0, T])$, where $\mathcal{F} = F_T$ and $T < \infty$. We work with discounted terms, i.e. the bond is assumed to be a constant.

Suppose that

$$S_t = M_t + \int_0^t \lambda_s \, d\langle M \rangle_s, \quad \int_0^t \lambda_s^2 \, d\langle M \rangle_s < \infty$$

for all t P-a.s., where M is a continuous local martingale and λ is a predictable process.

Let $\hat{U} = \hat{U}(x, \omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a random utility function, such that $\hat{U}$ is measurable and continuously differentiable, increasing, strictly concave P-a.s.. Let also be given the non-random utility function $U = U(x) : \mathbb{R}_+ \to \mathbb{R}$, tree-times continuously differentiable, increasing, strictly concave and $U'(\infty) = 0, \quad U'(0) = \infty$.

We denote by Π^x the set of admissible strategies with initial capital x which is now defined by

$$\Pi^x := \{\pi \text{ is predictable, } P(\int_0^T |\pi_s|^2 d\langle M \rangle_s < \infty) = 1\}.$$

The associated wealth process is given by $X_t^\pi := x + \int_0^t \pi_s X_s^\pi dS_s, \quad t \in [0, T]$. We consider the maximization problem

$$\max_{\pi \in \Pi_x} E\hat{U}(X_T^\pi)). \tag{1}$$

We furthermore need to impose the following assumptions on $\hat{U}$.

(B) There exists the maximizer (π^*, X^*) of problem (1),
i.e. $E\hat{U}(X_T^{\pi^*})) = \max_{\pi \in \Pi_x} E\hat{U}(X_T^\pi)), \ X^* = X^{\pi^*}$, and $\hat{U}'(X^*)$ is the density of the martingale measure.

Theorem 1. *Let condition B) be satisfied. Then*

$$\pi_t^* = -\frac{U_t'(X_t^*)}{X_t^* U_t''(X_t^*)} \left(\lambda_t + \frac{\psi_t}{P_t} \right),$$

where the triple (P,ψ,L), $L\perp M$ *is the solution of BSDE*

$$dP_t = P_t\Big[\Big(\lambda_t+\frac{\psi_t}{P_t}\Big)^2-\frac{1}{2}\frac{U_t'''(X_t^*)U_t'(X_t^*)}{U_t''(X_t^*)^2}\Big(\lambda_t+\frac{\psi_t}{P_t}\Big)^2\Big]d\langle M\rangle_t$$
$$+\psi_t dM_t + dL_t,\ P_T=\frac{\hat U'(X_T^*)}{U'(X_T^*)}.$$

Proof. It is easy to verify that $X_t^*\left(\int_0^t h_u(\lambda_u-\pi_u^*)d\langle M\rangle_u+\int_0^t h_u dM_u\right)$ is a local martingale for each bounded, predictable h w.r.t martingale measure. This means that $E[X_T^*\hat U'(X_T^*)|F_t]\left(\int_0^t h_u(\lambda_u-\pi_u^*)d\langle M\rangle_u+\int_0^t h_u dM_u\right)$ is a local martingale.
Let $P_t=\frac{E[X_T^*\hat U'(X_T^*)|F_t]}{X_t^*U'(X_t^*)}$. Then by the Ito formula we have decomposition

$$P_t = P_0+\int_0^t\alpha_u d\langle M\rangle_u+\int_0^t\psi_u dM_u+L_t. \tag{2}$$

Using the Ito formula for $F(X_t^*,P_t,H_t)=P_tX_t^*U'(X_t^*)H_t$, where $H_t=H_0+\int_0^t h_u(\lambda_u-\pi_u^*)d\langle M\rangle_u+\int_0^t h_u dM_u$, we obtain

$$\begin{aligned}
& dF(X_t^*,P_t,H_t)\\
&= \big[P_t(X_t^*U''(X_t^*)+U'(X_t^*))H_t\pi_t^*X_t^*\lambda_t+X_t^*U'(X_t^*)H_t\alpha_t\\
&+P_tX_t^*U'(X_t^*)(\lambda_t-\pi_t^*)h_t+P_t(U''(X_t^*)+\frac{1}{2}X_t^*U'''(X_t^*))H_t(\pi_t^*)^2(X_t^*)^2\\
&+U'(X_t^*)\psi_th_tX_t^*+(X_t^*U''(X_t^*)+U'(X_t^*))H_t\psi_t\pi_t^*X_t^*\\
&+(X_t^*U''(X_t^*)+U'(X_t^*))P_th_t\pi_t^*X_t^*\big]d\langle M\rangle_t+\text{local martingale}\\
&= \big\{H_t\big[P_t(X_t^*U''(X_t^*)+U'(X_t^*))\pi_t^*X_t^*\lambda_t+X_t^*U'(X_t^*)\alpha_t\\
&+P_t(U''(X_t^*)+\frac{1}{2}X_t^*U'''(X_t^*))(\pi_t^*)^2(X_t^*)^2+(X_t^*U''(X_t^*)+U'(X_t^*))\psi_t\pi_t^*X_t^*\big]\\
&+h_t\big[U'(X_t^*)\psi_tX_t^*\\
&+P_tX_t^*U'(X_t^*)(\lambda_t-\pi_t^*)+(X_t^*U''(X_t^*)+U'(X_t^*))P_t\pi_t^*X_t^*\big]\big\}d\langle M\rangle_t\\
&+\text{local martingale}
\end{aligned}$$

Equalazing coefficients of H and h to zero we get

$$\begin{aligned}
&P_t(X_t^*U''(X_t^*)+U'(X_t^*))\pi_t^*\lambda_t+U'(X_t^*)\alpha_t\\
&+P_t(U''(X_t^*)+\frac{1}{2}X_t^*U'''(X_t^*))(\pi_t^*)^2X_t^*+(X_t^*U''(X_t^*)+U'(X_t^*))\psi_t\pi_t^*=0
\end{aligned}$$

and

$$P_tU'(X_t^*)(\lambda_t-\pi_t^*)+U'(X_t^*)\psi_t+(X_t^*U''(X_t^*)+U'(X_t^*))P_t\pi_t^*=0.$$

Hence

$$\alpha_t = -\frac{(P_t\lambda_t + \psi_t)(X_t^* U''(X_t^*) + U'(X_t^*))}{U'(X_t^*)}\pi_t^* \tag{3}$$

$$-\frac{P_t(U''(X_t^*) + \frac{1}{2}X_t^* U'''(X_t^*))X_t^*}{U'(X_t^*)}|\pi_t^*|^2, \tag{4}$$

$$\pi_t^* = -\frac{U'(X_t^*)}{X_t^* U''(X_t^*)}\Big(\lambda_t + \frac{\psi_t}{P_t}\Big).$$

Plugging π^* into (3) yields

$$\begin{aligned}\alpha_t = \Big(\lambda_t + \frac{\psi_t}{P_t}\Big)P_t\Big[&\frac{(X_t^* U''(X_t^*) + U'(X_t^*))}{U'(X_t^*)}\frac{U'(X_t^*)}{X_t^* U''(X_t^*)} \\ &-\frac{U''(X_t^*) + \frac{1}{2}X_t^* U'''(X_t^*)}{U'(X_t^*)}\frac{U'(X_t^*)^2}{X_t^* U''(X_t^*)^2}\Big] \\ = \Big(\lambda_t + \frac{\psi_t}{P_t}\Big)&P_t\Big(1 - \frac{1}{2}\frac{U'''(X_t^*)U'(X_t^*)}{U''(X_t^*)^2}\Big),\end{aligned}$$

Therefore we get

$$dP_t = P_t\Big(\lambda_t + \frac{\psi_t}{P_t}\Big)^2\Big(1 - \frac{1}{2}\frac{U'''(X_t^*)U'(X_t^*)}{U''(X_t^*)^2}\Big)d\langle M\rangle_t + \psi_t dM_t + dL_t, \quad P_T = \frac{\hat{U}'(X_T^*)}{U'(X_T^*)}.$$

□

Corollary 1. *The quadruple* (X^*, Y, Z, N)*, where* $Y_t = \ln P_t,\ Z_t = \frac{\psi_t}{P_t},\ N_t = \int_0^t \frac{1}{P_s}dL_s$*, is the solution of the Forward Backward Stochastic Differential Equation (FBSDE)*

$$\begin{aligned} dX_t^* &= -\frac{U'(X_t^*)(\lambda_t + Z_t)}{U''(X_t^*)}dS_t,\ X_0^* = x, \\ dY_t &= \Big[(\lambda_t + Z_t)^2 - \frac{1}{2}\frac{U'''(X_t^*)U'(X_t^*)}{U''(X_t^*)^2}\Big(\lambda_t + Z_t\Big)^2\Big]d\langle M\rangle_t \\ &\quad -\frac{1}{2}Z_t^2 d\langle M\rangle_t - \frac{1}{2}d\langle N\rangle_t + Z_t dM_t + dN_t, \quad N \perp M, \\ Y_T &= \ln\frac{\hat{U}'(X_T^*)}{U'(X_T^*)}. \end{aligned}$$

Remark 1. The FBSDE for the case $\hat{U}(x) = U(x + H)$ was considered in [1] and [2]. We can show, that (X^*, Y), where $Y_t = (U')^{-1}(P_t U'(X_t)) - X_t^*$, is the solution of FBSDE (3.10) of [1]

R E F E R E N C E S

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