
Keywords and phrases: Banach space, permutation, rearrangement theorem, sum range, Steinitz range.


1 Introduction. The finite-dimensional version of the rearrangement theorem looks as follows:

**Theorem 1.** ([11], [5, Lemma I], [10]) Let $X$ be a finite-dimensional real normed space and let $(x_k)$ be a sequence of elements of $X$, $S_n = \sum_{k=1}^{n} x_k$, $n = 1, 2, \ldots$, and $S \in X$. If some subsequence $(S_{k_n})$ of $(S_n)$ converges in $X$ to $S$ and $(x_k)$ tends to 0 in $X$, then there exists a permutation $\pi$ of the set of natural numbers $\mathbb{N}$ such that the series $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges in $X$ to $S$.

The following statement represents a first extension of Theorem 1 to an infinite-dimensional case.

**Theorem 2.** [6, Lemma I] Let $1 \leq p < \infty$, $X = L^p$, $r = \min(p, 2)$ and $(x_k)$ be a sequence of elements of $X$, $S_n = \sum_{k=1}^{n} x_k$, $n = 1, 2, \ldots$, and $S \in X$. If some subsequence $(S_{k_n})$ of $(S_n)$ converges in $X$ to $S$ and $\sum_{k=1}^{\infty} \|x_k\|^r < \infty$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges in $X$ to $S$.

Later on Theorem 2 for $X = L^2$ was rediscovered in [2], where for the proof the following lemma is used:

**Lemma 1.** [2, Lemma] Let $H$ be a real Hilbert space, let $n > 1$ be a natural number, $x_k \in H$, $k = 1, \ldots, n$ and $a := \sum_{k=1}^{n} x_k$. Then it is possible to find a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that
\[
\| \sum_{j=1}^{k} x_{\pi(j)} \| \leq \| a \| + \left( \sum_{j=1}^{k} \| x_{\pi(j)} \|^{2} + \| a \| (\| a \| + 2M) \right)^{\frac{1}{2}}, \quad k = 1, \ldots, n,
\]
where \( M = \max_{1 \leq k \leq n} \| x_k \| \).

In [8] the following 'weak topology' version of [2, Theorem] is contained:

**Theorem 3.** [8, Theorem 1] Let \( H \) be a real Hilbert space, and let \( (x_k) \) be a sequence of elements of \( H \), \( S_n = \sum_{k=1}^{n} x_k \), \( n \in \mathbb{N} \), and \( S \in X \). If some subsequence \( (S_{k_n}) \) of \( (S_n) \) converges in the weak topology of \( H \) to \( S \) and \( \sum_{k=1}^{\infty} \| x_k \|^2 < \infty \), then there exists a permutation \( \pi : \mathbb{N} \to \mathbb{N} \) such that the series \( \sum_{k=1}^{\infty} x_{\pi(k)} \) converges in the weak topology of \( H \) to \( S \).

The proof of Theorem 3 in [8] is based on the following version of Lemma 1:

**Lemma 2.** [8, Lemma] Let \( H \) be a real Hilbert space with the inner product \( (\cdot | \cdot) \), let \( n > 1 \) be a natural number, \( x_k \in H \), \( k = 1, \ldots, n \) and \( a := \sum_{k=1}^{n} x_k \). Then it is possible to find a permutation \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that

\[
\left| \sum_{j=1}^{k} (x_{\pi(j)}|y) \right| \leq |(a|y)| + \left( \| y \|^{2} \sum_{j=1}^{k} \| x_{\pi(j)} \|^{2} + |(a|y)|(|(a|y)| + 2M_{y}) \right)^{\frac{1}{2}}, \quad k = 1, \ldots, n \quad \forall y \in H,
\]

where \( M_{y} = \max_{1 \leq k \leq n} |(x_k|y)| \).

In its turn in [8] Lemma 2 is derived from the following statement:

**Lemma 3.** [8, p.142] Let \( H \) be a real Hilbert space with the inner product \( (\cdot | \cdot) \), \( n > 1 \) be a natural number, \( x_k \in H \), \( k = 1, \ldots, n \) be such that \( \sum_{k=1}^{n} x_k = 0 \). Then it is possible to find a permutation \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that

\[
\left( \sum_{j=1}^{k} |(x_{\pi(j)}|y)| \right)^{2} \leq \sum_{j=1}^{k} |(x_{\pi(j)}|y)|^{2}, \quad k = 1, \ldots, n \quad \forall y \in H.
\]

As this is shown in [3], Lemma 3 is not correct. It follows that the proof of Theorem 3 presented in [8] is not correct as well. Nevertheless, we shall show that the following refined version of Theorem 3 is valid:

**Theorem 4.** Let \( H \) be a real Hilbert space, and let \( (x_k) \) be a sequence of elements of \( H \), \( S_n = \sum_{k=1}^{n} x_k \), \( n \in \mathbb{N} \), and \( S \in X \). If some subsequence \( (S_{k_n}) \) of \( (S_n) \) converges in the weak topology of \( H \) to \( S \) and \( \sum_{k=1}^{\infty} \| x_k \|^2 < \infty \), then there exists a permutation \( \pi : \mathbb{N} \to \mathbb{N} \) such that the series \( \sum_{k=1}^{\infty} x_{\pi(k)} \) converges in the topology of \( H \) to \( S \).
2 Steinitz’s range and proofs. For a topological vector space $X$ over $\mathbb{R}$ we write $X^*$ for the dual space consisting of all continuous linear functionals $x^* : X \to \mathbb{R}$.

To a sequence $(x_n)$ extracted from a Hausdorff (locally convex) topological vector space $X$ let us associate two subsets $SR(\sum_{k=1}^{\infty} x_k)$ and $StR(\sum_{k=1}^{\infty} x_k)$ of $X$ as follows:

- an element $x \in X$ belongs to $SR(\sum_{k=1}^{\infty} x_k)$ if there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the sequence $(\sum_{n=1}^{\infty} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in the topology of $X$ to $x$. The set $SR(\sum_{k=1}^{\infty} x_k)$ is called the sum range of $(x_n)$ (cf. [7, Definition 2.1.1]).

- an element $x \in X$ belongs to $StR(\sum_{k=1}^{\infty} x_k)$ if $x^*(x) \in SR(\sum_{k=1}^{\infty} x^*(x_k))$ for every $x^* \in X^*$. The set $StR(\sum_{k=1}^{\infty} x_k)$ is called the Steinitz range of $(x_n)$ [4].

For a sequence $(x_n)$ extracted from a Hausdorff locally convex topological vector space $X$ we have:

(a) $SR(\sum_{k=1}^{\infty} x_k) \subset StR(\sum_{k=1}^{\infty} x_k)$ (this is evident).

(b) If $StR(\sum_{k=1}^{\infty} x_k) \neq \emptyset$, then $StR(\sum_{k=1}^{\infty} x_k)$ is a (weakly) closed affine subspace of $X$ (this is essentially a consequence of Riemann’s theorem on conditionally convergent series, see [4, Proposition 2.1]; see also [1, Proposition 1], where this is proved under the assumption $SR(\sum_{k=1}^{\infty} x_k) \neq \emptyset$).

(c) If $X$ is finite-dimensional, then $SR(\sum_{k=1}^{\infty} x_k) = StR(\sum_{k=1}^{\infty} x_k)$ (in view of previous item it can be said that this is a reformulation of the famous Steinitz’s theorem on conditionally convergent series).

(d) Let $1 \leq p < \infty$ and $X = L^p$. Suppose that in case $2 \leq p < \infty$ we have $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$, while in case $1 < p \leq 2$ we have $\left(\sum_{k=1}^{\infty} x_k^2\right)^{1/2} \in X$, then $SR(\sum_{k=1}^{\infty} x_k) = StR(\sum_{k=1}^{\infty} x_k)$ [9, Theorem 1].

Proof of Theorem 4.

By [9, Theorem 1], it is sufficient to show that $S \in StR(\sum_{k=1}^{\infty} x_k)$. To do this, fix $y \in H$ and let us verify that

$$(S|y) \in SR\left(\sum_{k=1}^{\infty} (x_k|y)\right).$$

Clearly, some subsequence of the sequence $(\sum_{n=1}^{\infty} (x_k|y))_{n\in\mathbb{N}}$ converges to $(S|y)$; we have also that $\lim_{k}(x_k|y) = 0$. So, by one-dimensional version of Theorem 1 there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$(S|y) = \sum_{k=1}^{\infty} (x_{\pi(k)}|y).$$
Therefore,

\[(S|y) \in SR\left(\sum_{k=1}^{\infty} (x_k|y)\right)\]

From this, since \(y \in H\) is arbitrary, we get that \(S \in StR(\sum_{k=1}^{\infty} x_k)\).

**REFERENCES**


Received 17.05.2017; revised 30.09.2017; accepted 20.11.2017.

Author(s) address(es):

George Chelidze
I. Javakhishvili Tbilisi State University
Faculty of Exact and Natural Sciences
University str. 2, 0186 Tbilisi, Georgia
E-mail: giorgi.chelidze@tsu.ge

George Giorgobiani
Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University
Grigol Peradze str. 4, 0131, Tbilisi, Georgia
E-mail: giorgobiani.g@gtu.ge

Vaja Tarieladze
Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University
Grigol Peradze str. 4, 0131, Tbilisi, Georgia
E-mail: v.tarieladze@gtu.ge