Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

REARRANGEMENT THEOREM FOR THE WEAK TOPOLOGY *

George Chelidze George Giorgobiani Vaja Tarieladze

Abstract. We prove a refined version of a rearrangement theorem contained in B. K. Lahiri and S. K. Bhattacharya, *A note on rearrangements of series*, Math. Student **64**, 1-4 (1995), 141-145 (1996).

Keywords and phrases: Banach space, permutation, rearrangement theorem, sum range, Steinitz range.

AMS subject classification (2010): 22A05, 46A11.

1 Introduction.

The finite-dimensional version of *the rearrangement theorem* looks as follows:

Theorem 1. ([11], [5, Lemma I], [10]) *Let X be a finite-dimensional real normed space and let (x_k) be a sequence of elements of X, $S_n = \sum\limits_{k=1}^{n} x_k$, $n = 1, 2, \ldots,$ and $S \in X$. If some subsequence (S_{k_n}) of (S_n) converges in X to S and (x_k) tends to 0 in X, then there exists a permutation π of the set of natural numbers $\mathbb{N}$ such that the series $\sum\limits_{k=1}^{\infty} x_{\pi(k)}$ converges in X to S.*

The following statement represents a first extension of Theorem 1 to an infinite-dimensional case.

Theorem 2. [6, Lemma I] *Let $1 \le p < \infty$, $X = L^p$, $r = \min(p, 2)$ and (x_k) be a sequence of elements of X, $S_n = \sum\limits_{k=1}^{n} x_k$, $n = 1, 2, \ldots,$ and $S \in X$. If some subsequence (S_{k_n}) of (S_n) converges in X to S and $\sum\limits_{k=1}^{\infty} \|x_k\|^r < \infty$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series $\sum\limits_{k=1}^{\infty} x_{\pi(k)}$ converges in X to S.*

Later on Theorem 2 for $X = L^2$ was rediscovered in [2], where for the proof the following lemma is used:

Lemma 1. [2, Lemma] *Let H be a real Hilbert space, let $n > 1$ be a natural number, $x_k \in H, k = 1, \ldots, n$ and $a := \sum_{k=1}^{n} x_k$. Then it is possible to find a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

*Partially supported by Shota Rustaveli National Science Foundation project FR/539/5-100/13

$$\|\sum_{j=1}^{k} x_{\pi(j)}\| \le \|a\| + \left(\sum_{j=1}^{k} \|x_{\pi(j)}\|^2 + \|a\|(\|a\| + 2M)\right)^{\frac{1}{2}}, \ k = 1, \dots, n,$$

where $M = \max_{1 \le k \le n} \|x_k\|$.

In [8] the following 'weak topology' version of [2, Theorem] is contained:

Theorem 3. [8, Theorem 1] *Let H be a real Hilbert space, and let (x_k) be a sequence of elements of H, $S_n = \sum\limits_{k=1}^{n} x_k$, $n \in \mathbb{N}$, and $S \in X$. If some subsequence (S_{k_n}) of (S_n) converges in the weak topology of H to S and $\sum\limits_{k=1}^{\infty} \|x_k\|^2 < \infty$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series $\sum\limits_{k=1}^{\infty} x_{\pi(k)}$ converges in the weak topology of H to S.*

The proof of Theorem 3 in [8] is based on the following version of Lemma 1:

Lemma 2. [8, Lemma] *Let H be a real Hilbert space with the inner product $(\cdot|\cdot)$, let $n > 1$ be a natural number, $x_k \in H$, $k = 1, \dots, n$ and $a := \sum_{k=1}^{n} x_k$. Then it is possible to find a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ such that*

$$\left|\sum_{j=1}^{k} (x_{\pi(j)}|y)\right| \le |(a|y)| + \left(\|y\|^2 \sum_{j=1}^{k} \|x_{\pi(j)}\|^2 + |(a|y)|(|(a|y)| + 2M_y)\right)^{\frac{1}{2}}, \ k = 1, \dots, n \ \ \forall y \in H,$$

where $M_y = \max_{1 \le k \le n} |(x_k|y)|$.

In its turn in [8] Lemma 2 is derived from the following statement:

Lemma 3. [8, p.142] *Let H be a real Hilbert space with the inner product let $(\cdot|\cdot)$, $n > 1$ be a natural number, let $x_k \in H$, $k = 1, \dots, n$ be such that $\sum_{k=1}^{n} x_k = 0$. Then it is possible to find a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ such that*

$$\left|\sum_{j=1}^{k} (x_{\pi(j)}|y)\right|^2 \le \sum_{j=1}^{k} |(x_{\pi(j)}|y)|^2, \ k = 1, \dots, n \ \ \forall y \in H.$$

As this is shown in [3], Lemma 3 *is not correct*. It follows that the proof of Theorem 3 presented in [8] *is not correct as well*. Nevertheless, we shall show that the following refined version of Theorem 3 is valid:

Theorem 4. *Let H be a real Hilbert space, and let (x_k) be a sequence of elements of H, $S_n = \sum\limits_{k=1}^{n} x_k$, $n \in \mathbb{N}$, and $S \in X$. If some subsequence (S_{k_n}) of (S_n) converges in the weak topology of H to S and $\sum\limits_{k=1}^{\infty} \|x_k\|^2 < \infty$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series $\sum\limits_{k=1}^{\infty} x_{\pi(k)}$* **converges in the topology of** *H to S.*

2 Steinitz's range and proofs. For a topological vector space X over $\mathbb{R}$ we write X^* for the dual space consisting of all continuous linear functionals $x^* : X \to \mathbb{R}$.

To a sequence (x_n) extracted from a Hausdorff (locally convex) topological vector space X let us associate two subsets $SR\,(\sum_{k=1}^{\infty} x_k)$ and $StR\,(\sum_{k=1}^{\infty} x_k)$ of X as follows:

- an element $x \in X$ belongs to $SR(\sum_{k=1}^{\infty} x_k)$ if there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the sequence $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in the topology of X to x. The set $SR(\sum_{k=1}^{\infty} x_k)$ is called *the sum range* of (x_n) (cf. [7, Definition 2.1.1]).
- an element $x \in X$ belongs to $StR(\sum_{k=1}^{\infty} x_k)$ if $x^*(x) \in SR\,(\sum_{k=1}^{\infty} x^*(x_k))$ for every $x^* \in X^*$. The set $StR(\sum_{k=1}^{\infty} x_k)$ is called *the Steinitz range* of (x_n) [4].

For a sequence (x_n) extracted from a Hausdorff locally convex topological vector space X we have:

(a) $SR(\sum_{k=1}^{\infty} x_k) \subset StR(\sum_{k=1}^{\infty} x_k)$ (this is evident).

(b) If $StR(\sum_{k=1}^{\infty} x_k) \neq \emptyset$, then $StR(\sum_{k=1}^{\infty} x_k)$ is a (weakly) closed affine subspace of X (this is essentially a consequence of Riemann's theorem on conditionally convergent series, see [4, Proposition 2.1]; see also [1, Proposition 1], where this is proved under the assumption $SR(\sum_{k=1}^{\infty} x_k) \neq \emptyset$).

(c) If X is finite-dimensional, then $SR(\sum_{k=1}^{\infty} x_k) = StR(\sum_{k=1}^{\infty} x_k)$ (in view of previous item it can be said that this is a reformulation of the famous Steinitz's theorem on conditionally convergent series).

(d) Let $1 \leq p < \infty$ and $X = L^p$. Suppose that in case $2 \leq p < \infty$ we have $\sum\limits_{k=1}^{\infty} \|x_k\|^2 < \infty$, while in case $1 < p \leq 2$ we have $\left(\sum\limits_{k=1}^{\infty} x_k^2\right)^{\frac{1}{2}} \in X$, then $SR(\sum_{k=1}^{\infty} x_k) = StR(\sum_{k=1}^{\infty} x_k)$ [9, Theorem 1].

Proof of Theorem 4.

By [9, Theorem 1], it is sufficient to show that $S \in StR(\sum_{k=1}^{\infty} x_k)$. To do this, fix $y \in H$ and let us verify that

$$(S|y) \in SR\left(\sum_{k=1}^{\infty}(x_k|y)\right).$$

Clearly, some subsequence of the sequence $(\sum_{k=1}^{n}(x_k|y))_{n\in\mathbb{N}}$ converges to $(S|y)$; we have also that $\lim_k (x_k|y) = 0$. So, *by one-dimensional version of Theorem 1* there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$(S|y) = \sum_{k=1}^{\infty}(x_{\pi(k)}|y).$$

Therefore,

$$(S|y) \in SR\left(\sum_{k=1}^{\infty}(x_k|y)\right)$$

From this, since $y \in H$ is arbitrary, we get that $S \in StR(\sum_{k=1}^{\infty} x_k)$.

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Received 17.05.2017; revised 30.09.2017; accepted 20.11.2017.

Author(s) address(es):

George Chelidze
I. Javakhishvili Tbilisi State University
Faculty of Exact and Natural Sciences
University str. 2, 0186 Tbilisi, Georgia
E-mail: giorgi.chelidze@tsu.ge

George Giorgobiani
Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University
Grigol Peradze str. 4, 0131, Tbilisi, Georgia
E-mail: giorgobiani.g@gtu.ge

Vaja Tarieladze
Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University
Grigol Peradze str. 4, 0131, Tbilisi, Georgia
E-mail: v.tarieladze@gtu.ge