Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

ADAPTIVE MULTISCHEME REFINEMENT FOR LINEAR ADVECTION EQUATION ON CARTESIAN MESHES IN TWO SPACE DIMENSIONS. *

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Abstract. Multischeme is a smart combination of numerical schemes and meshes. The approach is developed for time dependent linear advection equation in two space dimensions. Monotone and MUSCL schemes are coupled with mesh refinement. Convergence of multischemes is proved. Numerical tests demonstrate efficiency of the developed approach.

Keywords and phrases: Finite volumes, mesh adaptation, monotone numerical flux, slope limiters, linear advection.

AMS subject classification (2010): 65M08, 65M50.

1 Introduction. The initial value problem for the linear advection equation in two space dimensions in Cartesian coordinates is written

$$\partial u/\partial t + \nabla \cdot (u\overrightarrow{v}) = 0, u(0,x,y) = u_0(x,y), \tag{1}$$

where $u = u(t,x,y), v = (v_1(t,x,y), v_2(t,x,y))^T, t > 0, (x,y) \in R^2$; $u_0(x,y), v(t,x,y)$ are assumed to be sufficiently smooth functions that ensure the solution $u(t,x,y)$ is also sufficiently smooth. One of the most important applications of this equation is in environmental modelling where $\overrightarrow{v}$ is interpreted as velocity vector and u is a concentration of some species, see e.g. [3]. Fast and accurate numerical methods resulting in saving of computational cost in terms of computing time and computer memory is of great interest when size of computational domain is big and numerical solution with high accuracy is needed. Here we propose one of such methods - multischeme for (1). It is extension of the one dimensional multischeme method for scalar conservation laws [2]. Efficiency of multischeme is achieved by coupling of numerical schemes of different accuracy and meshes of different space and time resolution. Explicit first order monotone and high resolution MUSCL finite volume schemes are considered on uniform Cartesian meshes.

2 Monotone fluxes, slope limiters. Finite volume schemes including first order monotone and high resolution MUSCL schemes are well documented in the literature [4], [5]. Standard explicit finite volume discretization of the equation (1) on uniform Cartesian mesh with time step Δt, space steps Δx and Δy in x and y directions respectively, is written:

$$(u_{ij}^{n+1} - u_{ij}^n)/\Delta t + (F_{i+1/2,j}^n - F_{i-1/2,j}^n)/\Delta x + (G_{i,j+1/2}^n - G_{i,j-1/2}^n)/\Delta y = 0, \tag{2}$$

*This work was partially supported by the TSU project “Multischemes in two space dimensions”.

where $F_{i\mp 1/2,j}, G_{i,j\mp 1/2}$ are numerical fluxes approximating uv_1 and uv_2 at $(t_n, x_{i\mp 1/2,j}, y_j)$ and $(t_n, x_i, y_{j\mp 1/2})$ respectively. The scheme (2) is first order accurate in time in the sense of local truncation error. Numerical fluxes determine properties of the numerical scheme (2) such as truncation error due to spatial discretization, conservativeness, stability, convergence. Here we consider monotone numerical fluxes

$$F^n_{i+1/2,j} = F(u^n_{i+1,j}, u^n_{i,j}, v^n_{1,i+1,j}, v^n_{1,i,j}), G^n_{i,j+1/2} = G(u^n_{i,j+1}, u^n_{i,j}, v^n_{2,i,j+1}, v^n_{2,i,j}),$$

where F and G are at least Lipschitz-continuous functions consistent with fluxes of the equation (1), i.e. uv_1 and uv_2; they are also monotone, i.e. non increasing with respect to the first and nondecreasing with respect to the second variable. One of the examples is Engquist-Osher type flux splitting defined by the following formulas

$$F(u, w, v_{1u}, v_{1w}) = uv_{1u} + wv_{1w}, G(u, w, v_{2u}, v_{2w}) = uv_{2u} + wv_{2w},$$
$$v_{iu} = 0.5(v_i - |v_i|), v_{iw} = 0.5(v_i + |v_i|), i = 1, 2.$$

Monotone schemes are first order accurate. For achieving higher order high resolutuion MUSCL schemes are used based on slope limiters [6] , e.g. defined by

$$F^n_{i+1/2,j} = F(u^n_{i+1^-,j}, u^n_{i^+,j}, v^n_{1,i+1^-,j}, v^n_{1,i^+,j}), G^n_{i,j+1/2} = G(u^n_{i,j+1^-}, u^n_{i,j^+}, v^n_{2,i,j+1^-}, v^n_{2,i,j^+}),$$
$$u^n_{i^\pm,j} = u^n_{i,j} \pm 0.5\Delta^x_{i,j}(u^n), v^n_{1,i^\pm,j} = v^n_{1,i,j} \pm 0.5\Delta^x_{i,j}(v^n_1),$$
$$u^n_{i,j^\pm} = u^n_{i,j} \pm \Delta^y_{i,j}(u^n), v^n_{2,i,j^\pm} = v^n_{2,i,j} \pm \Delta^y_{i,j}(v^n_2),$$

Δ^x and Δ^y are so called slope limiters, e.g. $\Delta^x_{i,j}(w^n) = minmod(w^n_{i+1,j} - w^n_{i,j}, w^n_{i,j} - w^n_{i-1,j}), \Delta^y_{i,j}(u^n) = minmod(w^n_{i,j+1} - w^n_{i,j}, w^n_{i,j} - w^n_{i,j-1})$; where $minmod(a, b)$ is 0 if $ab < 0$ otherwise it returns the argument with smallest modulus.

3 Constructing multischeme: mesh refinement and coupling of schemes. Strtuctured adaptive mesh refinement algorithm for hyperbolic problems consists of the fololwing three steps[1]: *1.compute solution on entire coarse grid; 2. compute solution on entire fine grid; 3. sinchronize coarse and fine grid solutions.* Synchronization usually contains procedures of averaging, interpolation and correction of solutions in cells adjacent to different meshes, see e.g. [1]. Multischeme can be interpreted in the framework of the above general algorithm if first and second steps are exchanged and if synchronization is defined as computing of solution near interfaces of space-time meshes of different resolution. Construction of multischeme is simple since it is usual finite volume discretization of the equation (1) on space time unstructured mesh using different numerical flux functions. Then general algorithm for multischeme construction is the following: *1. define consistent numerical flux function for each cell interface; 2. perform finite volume discretization using numerical fluxes defined in the step one.* Finding optimal mesh refinement is not trivial task, it is difficult to assess if efforts spent on optimal meshing will pay out. Though it is well known that mesh size, gradients and accuracy of computations are correlated. It is also well known that accurate resolution of big gradients requires fine mesh; finer meshes are also needed near local extrema points since MUSCL schemes are reduced to the first order there. Therefore we combine gradients and local extrema into the following simple mesh refinement and coarsening criteria: *1.refine cells near local extremas or if gradient is out of predefined interval; 2. merge cells if they do not contain local extremas and if gradients is in predefined interval.* Multischeme uses higher order MUSCL scheme in some nodes and lower order monotone schemes in others. We apply higher order schemes for refined meshes only, see numerical tests. With multischemes one can not perform computations in arbitrary order but certain rules, illustrated in figure 1, should be respected.

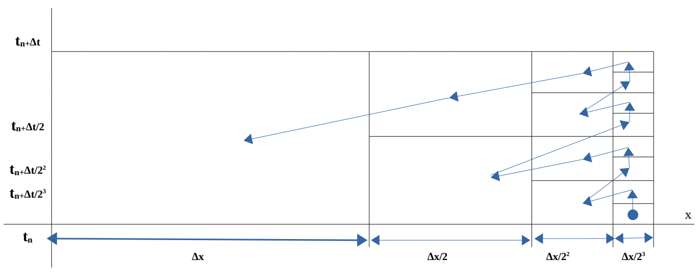

Figure 1: Order of time marching

4 Properties of multischeme. It is well known that monotone schemes and MUSCL schemes with sufficiently smooth numerical fluxes are at least first order accurate in the sense of local truncation error. Under Courant-Friedrichs-Levy(CFL) condition they also ensure uniform boundedness of approximate solutions on any finite time interval. Therefore multischeme for the linear advection equation has the same properties since it can be written in the form of 2 where depending on nodal point's location numerical flux coincides with the one of monotone scheme, with the one of MUSCL scheme or it can be represented as some linear combination of these fluxes. For the proof of convergence it remains to apply Lax's equivalence theorem and we have the following

Theorem 1. *Suppose in (1) u_0, v_1, v_2 are sufficiently smooth functions and CFL condition is satisfied. Then multischeme with monotone numerical fluxes and MUSCL slope limiters is convergent.*

5 Numerical tests. Test problem - moving pyramid with constant speed along the diagonal is selected for comparing multischeme with other schemes; the pyramid is defined by initial function $u_0(x, y)$ and velocity vector is constant. Calculations of this test are done by monotone and MUSCL schemes on uniform meshes, as well as with monotone scheme coupled with mesh refinement and multischeme. Courant-Friedrichs-Levy number is 0.75 for all calculations. Some results are given on the figure 2. Results of calculations in terms of number of nodes and norms of errors are summarized in table 1.

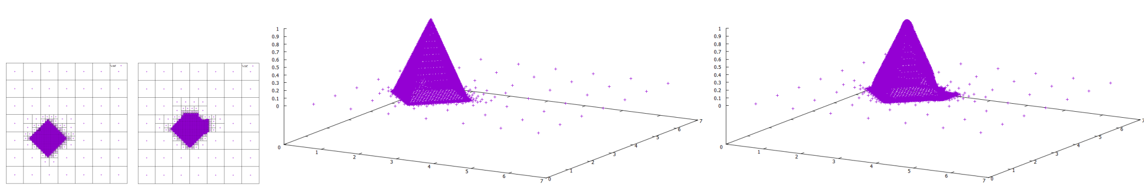

Figure 2: Refined meshes and pyramids at two different time moments

scheme	refinement	Nbr.nodes	err_∞	$err_{\infty,rel}$	err_1	$err_{1,rel}$
monotone	0,$\Delta/2^3$	3136	0.196258	0.206676	0.129533	0.00301262
MUSCL	0,$\Delta/2^3$	3136	0.121181	0.127614	0.070919	0.0016494
monotone	3	691	0.573156	0.619628	0.534242	0.0125563
monotone	0,$\Delta/2^6$	200704	0.0228918	0.0230713	0.00272841	$0.99 * 10^{-6}$
MUSCL	0,$\Delta/2^6$	200704	0.0135528	0.013659	0.00179124	$0.65 * 10^{-6}$
monotone	6	15574	0.234649	0.237619	0.134728	$0.49 * 10^{-6}$
mulstischeme	6, > 3	10717	0.1393	0.141064	0.0487799	$0.17 * 10^{-6}$

Table 1: Multischeme vs monotone and MUSCL schemes on uniform and adaptive meshes; no mesh adaptation if refinement=0, for multischeme finest refinement level is 6 and MUSCL is used when refinement > 3. Multischeme gives better accuracy with significantly smaller number of nodes.

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Received 10.06.2017; revised 15.09.2017; accepted 30.11.2017.

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