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ON THE APPLICATION OF BERNSTEIN TYPE CONSTRUCTION TO MEASURE EXTENSION PROBLEM *

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Abstract. We consider some properties of functions, which have thick (or massive) graphs with respect to certain classes of measure and some applications of set-theoretical and algebraic methods to measurability of sets and function.

Keywords and phrases: Absolutely nonmeasurable function, relative measurable function, λ-thick graph, extension of function.

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1 Introduction. The historical roots of measure theory have three fundamental notions:

(a) the length of interval;

(b) the area of a region;

(c) the volume of a solid.

The development of mathematics made it necessary to generalize these notions, so that more complicated subsets of given spaces (the real line $\mathbf{R}$, the plane $\mathbf{R^2}$, the space $\mathbf{R^3}$, or in general, n-dimensional Euclidean space $\mathbf{R^n}$) could be equipped with sigma-finite measures.

A partial solution this problems of finding a notion of a measure was the so-called Jordan measure, constructed by Peano and Jordan, though the collection, just described, of Jordan measurable sets seems large and is indeed large enough from the point of view of the Riemann integral theory, it is easy to see that it does not even contain all open subsets of same space. Lebesgue invented a different notion of a measure, widely accepted today as the ultimate solution to the problem of finding a satisfactory way to measure complicated sets. In contrast to the Jordan measure, the approximation figures may now also consist of countable many elementary areas and the most important new feature of the lebesgue measure is its countable additive. It is well-known that the Lebesgue measure is a proper extension of the Jordan measure (see. [5]).

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2 Content. Let E be a set and let M be a class of measures on E (in general, we do not assume that measures belonging to M are defined on the one and same σ-algebra of subset of E). The following definition is essentially due to E. Marczewski (see. [8]).

Definition 1. We say that a function $f : E \to \mathbf{R}$ is absolutely (or universally) measurable with respect to M if f is measurable with respect to all measures from M.

Definition 2. We say that a function $f : E \to \mathbf{R}$ is relatively measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is μ-measurable.

Definition 3. We recall that a subset X of $\mathbf{R^2}$ is λ_2-thick (or λ_2 massive) in $\mathbf{R^2}$ if for each λ_2-measurable set $Z \subseteq \mathbf{R^2}$ with $\lambda_2(Z) > 0$, we have

$$X \cap Z \neq \emptyset.$$

In other words, X is λ_2-thick in $\mathbf{R^2}$ if and only if the equality

$$(\lambda_2)_*(\mathbf{R^2} \setminus \mathbf{X}) = \mathbf{0}$$

is satisfied.

Let us briefly describe one useful method of extending measures by applying those mappings whose graphs are thick from the measure-theoretical point of view. This method was introduced by Kodaira and Kakutani in their famous construction of a nonseparable translation-invariant extension of Lebesgue measure on $\mathbf{R}$. (see. [5],[6],[7]).

Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be measurable spaces equipped with sigma-finite measures. We recall that a graph $\Gamma \subset E_1 \times E_2$ is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$ if for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$ with $(\mu_1 \times \mu_2)(Z) > 0$, we have $\Gamma \cap Z \neq 0$.

Notice that, the thickness of graphs is a pathological phenomenon for subsets of the basic set. However, this feature plays an essential role in the problem of extensions of measures.(see. [3],[4], [5])

Theorem 1. *Let E_1 be a set equipped with a sigma-finite measure μ and let $f : E_1 \to E_2$ be a function satisfying the following condition: there exists a probability measure μ_2 on $ran(f)$ such that the graph of f is a $(\mu_1 \times \mu_2)$-thick of the product set $E_1 \times ran(f)$. Then there exists the measure μ' such that:*

1) μ' is measure extending μ_1;

2) f is measurable with respect to μ'.

Theorem 2. *Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be two uncountable sets equipped with sigma-finite measures and $card(E_1) = card(E_2) = \alpha$. Suppose that there exists a family $\{Z_\xi : \xi < \alpha\}$ of subsets of $E_1 \times E_2$ satisfying the following conditions:*

(1) for any $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$, with $(\mu_1 \times \mu_2)(Z) > 0$, there is an index $\xi < \alpha$ such that $Z_\xi \subset Z$;

(2) for any $\xi < \alpha$, we have $card(pro_1 Z_\xi) = \alpha$.

Then there exists a function $f : E_1 \to E_2$ whose graph is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$.

Example 1. Any function, which has a λ_2-massive graphic, is relative measurable with respect to the class of extensions of Lebsgue measure;

3 Compilation.

We would like to present the purely algebraic method of extension of measures so-called Kharazishvili method of surjective homomorphisms. (see.[3],[4],[5]). In this context, note that the method of direct products, which is often applied in the theory of invariant or quasi-invariant measures, is a particular case of the method of surjective homomorphisms. These two methods can be described as follows.

Let (G_1, μ_1) and (G_2, μ_2) be any two groups endowed with σ-finite invariant measures and let

$$f : G_1 \to G_2$$

be a surjective homomorphism. Suppose that a general property $T(X)$ of a set $X \subset G_2$ is given. Sometimes, it turns out that

$$T(f^{-1}(X)) \Leftrightarrow T(X).$$

In such a situation we say that $T(X)$ is stable under surjective homomorphisms.

In particular, If f is a canonical surjective homomorphism

$$pr_2 : H \times G_2 \to G_2,$$

then we may apply the method of direct products, where the role of G_1 is played by $H \times G_2$.

4 Conclusions.

We have considered the λ_2-thickness of the functions and the method of Bernstein type constructions. In particular, by using a Bernstein type transfinite argument a function $f : \mathbf{R} \to \mathbf{R}$ can be constructed, whose graph is thick with respect to the product measure $\mu_1 \bigotimes \mu_2$ and, consequently f is nonmeasurable with respect to λ.

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