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ON A DIRICHLET PROBLEM FOR ONE SIXTH ORDER ELLIPTIC EQUATION

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Abstract. The Dirichlet problem for the sixth order properly elliptic equation is considered. We suppose that the corresponding characteristic equation has three different roots: imaginary unit with multiplicity two, one simple root with positive imaginary part, and triple root with negative imaginary part. The formula for this problem's defect numbers is obtained, and it is proved, that the defect numbers may only have values zero and one. The linearly independent solutions of homogeneous Dirichlet problem and the linearly independent solvability conditions of the inhomogeneous problem are determined in an explicit form.

Keywords and phrases: Dirichlet problem, defect numbers, nontrivial solutions of homogeneous Dirichlet problem, properly elliptic equation.

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1 Introduction. Let $D = \{z : z = x + iy, |z| < 1\}$ be the unit disk of the complex plane and $\Gamma = \partial D$. In the D we consider sixth order properly elliptic equation

$$\frac{\partial^2}{\partial \bar{z}^2}\left(\frac{\partial}{\partial \bar{z}} - \mu\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z} - \nu\frac{\partial}{\partial \bar{z}}\right)^3 u(x,y) = 0, \qquad (x,y) \in D, \tag{1}$$

where μ and ν are such complex constants that $0 < |\mu| < 1, \;\; 0 < |\nu| < 1$ (here $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are the operators of complex differentiation, $\bar{z}$ is a complex conjugate of z). We consider the Dirichlet problem for the equation (1) in a classical formulation, that is the solution to be found u belongs to the class $C^6(D) \bigcap C^{(2,\alpha)}(\overline{D})$ and on the boundary Γ satisfies the Dirichlet conditions

$$\left.\frac{\partial^j u}{\partial r^j}\right|_{\Gamma} = f_j(x,y), \quad j = 0,1,2; \quad (x,y) \in \Gamma. \tag{2}$$

Here $f_j \in C^{(2-j,\alpha)}(\Gamma)$ are the given functions. It is known, that the problem (1), (2) is Fredholmian (see, for example, [1], [2]). The general formula, for the determination defect numbers of the problem for arbitrary order of equation (1) was found in [3]. After that it was shown, that more exact results may be obtained. In [4] the fourth order equation was investigated, and it was proved, that in this case the defect numbers can not exceed one. In the paper we consider the sixth order equation.

2 Main result. The main result of the article is the following

Theorem 1. *Let $\zeta = \mu\nu$. Then the problem (1), (2) is the uniquely solvable if and only if the conditions*

$$P_n(\zeta) = \sum_{k=0}^{n} (k+1)(k+2)\zeta^k \neq 0, \quad n = 1, 2, \ldots \tag{3}$$

hold. If the conditions (3) fail for some n_0 *then the homogeneous problem (1), (2) (when* $f_j \equiv 0$ *for* $j = 0, 1, 2$*) has one nontrivial solution, which is a polynomial of order* $n_0 + 5$*, and one condition on boundary functions is necessary and sufficient for the solvability of the inhomogeneous problem. The conditions (3) may fail only for one* $n \geq 1$*, and, therefore, the defect numbers of the problem (1), (2) may only be equal to zero and one.*

Proof. The general solution of the equation (1) may be represented in the form ([3]):

$$u(z, \bar{z}) = \Phi_1(z) + \bar{z}\Phi_2(z) + \Phi_3(z + \mu\bar{z}) + \sum_{j=1}^{3} \left(\frac{\partial}{\partial\theta}\right)^{j-1} \Psi_j(\bar{z} + \nu z), \tag{4}$$

where Φ_1, Φ_2 are analytic functions in D, Φ_3 and Ψ_j, $j = 1, 2, 3$, are the functions analytic in the domains $D_1 = \{z + \mu\bar{z} : z \in D\}$ and $D_2 = \{\bar{z} + \nu z : z \in D\}$ respectively, $\frac{\partial}{\partial\theta}$ is a differentiation by the argument, $z = re^{i\theta}$. Let's represent the boundary conditions (2) in the equivalent (see [3]) form:

$$\left.\frac{\partial^2 u}{\partial z^j \partial \bar{z}^{2-j}}\right|_\Gamma = F_j(\theta), \quad j = 0, 1, 2; \quad (x, y) \in \Gamma, \quad z = x + iy = re^{i\theta}, \tag{5}$$

and

$$u(1, 0) = f_0(1, 0), \quad u_r(1, 0) = f_1(1, 0), \quad u_\theta(1, 0) = f_{0\theta}(1, 0). \tag{6}$$

where $F_j \in C^{(\alpha)}(\Gamma)$ are the functions uniquely determined by the boundary functions f_j. For the determination of unknown functions Φ_j and Ψ_j let's substitute (4) in the boundary conditions (5). Using the operator identity ([3]) $\frac{\partial^{k+m}}{\partial z^k \partial \bar{z}^m} \left(\frac{\partial}{\partial\theta}\right)^l = \left(\frac{\partial}{\partial\theta} + (k-m)I\right)^l \frac{\partial^{k+m}}{\partial z^k \partial \bar{z}^m}$, we get:

$$\mu^2 \Phi_3''(z + \mu\bar{z}) + \sum_{j=1}^{3} \left(\frac{\partial}{\partial\theta} - 2i\right)^{j-1} \Psi_j''(\bar{z} + \nu z) = F_0(\theta), \quad z = e^{i\theta}, \tag{7}$$

$$\Phi_2'(z) + \mu\Phi_3''(z + \mu\bar{z}) + \nu \sum_{j=1}^{3} \left(\frac{\partial}{\partial\theta}\right)^{j-1} \Psi_j''(\bar{z} + \nu z) = F_1(\theta), \quad z = e^{i\theta}, \tag{8}$$

$$\Phi_1''(z) + \bar{z}\Phi_2''(z) + \Phi_3''(z + \mu\bar{z}) + \nu^2 \sum_{j=1}^{3} \left(\frac{\partial}{\partial\theta} + 2i\right)^{j-1} \Psi_j''(\bar{z} + \nu z) = F_2(\theta), \quad z = e^{i\theta}. \tag{9}$$

We use representation of the unknown functions from (4) on the boundary Γ ([2], 189-190 pages):

$$\Phi_j(z) = \sum_{k=0}^{\infty} A_{jk} z^k, \ j = 1, 2; \ \Phi_3''(z + \mu\bar{z}) = \varphi_3(z) + \varphi_3(\mu\bar{z}) = \sum_{k=0}^{\infty} A_{3k} z^k + \sum_{k=0}^{\infty} A_{3k} \left(\frac{\mu}{z}\right)^k,$$

$$\Psi_j''(\bar{z}+\nu z)=\psi_j(\bar{z})+\psi_j(\nu z)=\sum_{k=0}^{\infty}B_{jk}z^{-k}+\sum_{k=0}^{\infty}B_{jk}\nu^k z^k,\quad z=e^{i\theta},\quad j=1,2,3. \tag{10}$$

The unknown functions Φ_3'' and Ψ_j'' are uniquely determined by the analytic in D functions φ_3 and ψ_j, therefore the problem (1), (2) is reduced to determination of the Taylor coefficients A_{jk} and B_{jk}. For this goal we substitute the expansions (10) and Fourier expansions for the functions F_j, $F_j(\theta)=\sum_{k=-\infty}^{\infty}d_{jk}z^k$, $j=0,1,2$ $z=e^{i\theta}$ in equations (7)-(9). We must mention, that the equalities (10) may be differentiated by θ, because (10) holds for all $z\in\Gamma$. Substituting in (7), we get:

$$\sum_{k=0}^{\infty}A_{3k}\mu^2 z^k+\sum_{k=0}^{\infty}A_{3k}\mu^{k+2}z^{-k}$$

$$+\sum_{j=1}^{3}\left(\sum_{k=0}^{\infty}B_{jk}(-ik-2i)^{j-1}z^{-k}+\sum_{k=0}^{\infty}B_{jk}(ik-2i)^{j-1}\nu^k z^k\right)=\sum_{k=-\infty}^{\infty}d_{0k}z^k,\ |z|=1,$$

and similar equalities for (8) and (9). Equating the coefficients of the same powers of z and $\bar{z}$, for $k\geq 1$ we get a system for the determination A_{3k} and B_{jk}:

$$\begin{gathered}\mu^{k+2}A_{3k}+B_{1k}-i(k+2)B_{2k}-(k+2)^2B_{3k}=d_{-0k},\\ \mu^{k+1}A_{3k}+\nu B_{1k}-i\nu kB_{2k}-\nu k^2B_{3k}=d_{-1k},\\ \mu^kA_{3k}+\nu^2B_{1k}-i\nu^2(k-2)B_{2k}-\nu^2(k-2)^2B_{3k}=d_{-2k},\\ \mu^2A_{3k}+\nu^kB_{1k}+i\nu^k(k-2)B_{2k}-\nu^k(k-2)^2B_{3k}=d_{0k}.\end{gathered} \tag{11}$$

First, we consider the system (11) for $k\geq 3$. The determinant of this system is equal to $\Delta_k=4i\nu\zeta^2(1-\zeta)^3k(k-1)(k-2)P_{k-3}(\zeta)$. Supposing that the conditions (3) hold, we can uniquely determine the coefficients A_{3k} and B_{jk} for $k\geq 3$. For $k=0,1,2$ appropriate coefficients may be found for arbitrary boundary functions, but not uniquely, and after that, from (8) and (9), we will find Φ_j $(j=1,2)$. Hence, in this case inhomogeneous problem (1), (2) has a solution, and the solution of the corresponding homogeneous problem is a polynomial of order at most five, but the homogeneous conditions (2) imply, that non-zero polynomial solution of homogeneous problem (1), (2) is divisible by $(1-z\bar{z})^3$([5], theorem 5.1). Therefore, the problem (1), (2) is uniquely solvable. And, at last, taking into account, that $\Delta_k\to 16i\nu z^2\neq 0$ for $k\to\infty$, we see that the coefficients A_{jk} and B_{jk} have the same rate of growth in infinity as d_{jk}, so the functions Φ_{jk}'', Ψ_{jk}'' satisfy the Holder condition in the closed disc D. It means, that the solution u of the problem (1), (2) belongs to the prescribed class $C^{(2,\alpha)}(\overline{D})$. First part of the theorem is proved.

Now, let's suppose, that for some $\zeta=\mu\nu$ the condition (3) failed for m, that is $\Delta_{m+3}=0$ $(m\geq 1)$. In this case homogeneous problem (1), (2) has non-zero solution, which is polynomial of order $m+5$ (for example, if $\zeta=\mu\nu=-\frac{1}{3}$ then $P_1(\zeta)=0$, and, therefore, $\Delta_4=0$. In this case, the polynomial $(1-z\bar{z})^3$ is non-trivial solution of the homogeneous problem (1), (2)), and one solvability condition is neccessary for the solvability of the inhomogeneous problem. Let ζ be the root of n-th order polynomial (3), that is $P_n(\zeta) = 0$. Then, by the Enestrom-Kakeya theorem (see [6], corollary 8.3.5) ζ lies in the ring $T_1 = \{z : \frac{1}{3} \leq |z| \leq \frac{n}{n+2}\}$. If we suppose, that for some $m > 0$ $P_{n+m}(\zeta) = 0$, then ζ will be the root of the polynomial $Q_{m-1}(t) = t^{-n-1}(P_{n+m}(t) - P_n(t)) = \sum_{l=0}^{m-1}(l + n + 2)(l + n + 3)t^l$, and, therefore, by the same Enestrom-Kakeya theorem belongs to the ring $T_2 = \{z : \frac{n+2}{n+4} \leq |z| \leq \frac{n+m}{n+m+2}\}$. But this is impossible, because the rings T_1 and T_2 are disjoint. Thus, for arbitrary ζ there is at most one value $n > 0$, for which $P_n(\zeta) = 0$, therefore, the defect numbers of the problem (1), (2) may only be equal to zero or one. Theorem is proved. $\square$

3 Conclusion. As was shown earlier, for second order equation (1), the Dirichlet problem (1), (2) is uniquely solvable. For fourth order equation in all known cases the defect numbers of this problem do not exceed one. The same result we get for some six order equations (1). So, it will be interesting to check the following hypothesis: the defect numbers of the Dirichlet problem for the properly elliptic equation (1) of order $2n$ do not exceed $n - 1$.

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