BASIC GROUPS OPERATIONS ON EXPONENTIAL $MR$-GROUPS

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Abstract. In the paper it is proved that the tensor completion is commutative with the operations of direct product and direct limit of exponential $MR$-groups and, but in general, is not commutative with the Cartesian product and the inverse limit of exponential $MR$-groups.

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The notion of an exponential $R$-group ($R$ is an arbitrary associative ring with identity) was introduced by R. Lyndon in [1]. In [2] A. G. Myasnikov refined the notion of an exponential $R$-group by introducing an additional axiom. In particular, the new notion of exponential $R$-group is a direct generalization of the notion of a $R$-module to the case of noncommutative groups. With in honour to A. G. Myasnikov, $R$-group this axiom in M. Amaglobeli’s paper [3] has homed $MR$-groups. Systematic study of $MR$-groups has started in [4, 5, 6, 7, 8, 9]. Note that the results of the study were very useful for solving the well-known problems of Tarski. In paper [2] it is shown that in the investigation of exponential $MR$-group the decisive role is played by the notion of tensor completion. In this paper we investigate the problem of the commutability of a functor of tensor completion with basic groups operations.

1 Basic notions in the theory of exponential $MR$-groups. Recall the basic definitions (see [1, 2]). Let $R$ be an arbitrary associative ring with identity and let $G$ be a group. Fix an action of the ring $R$ on $G$, i.e. a map $G \times R \to G$. The result of the action of $\alpha \in R$ on $g \in G$ is written as $g^\alpha$. Consider the following axioms:

(i) $g^1 = g$, $g^0 = e$, $e^\alpha = e$ ($1 \in R$, $e \in G$);
(ii) $g^{\alpha + \beta} = g^\alpha \cdot g^\beta$, $g^{\alpha \beta} = (g^\alpha)^\beta$;
(iii) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
(iv) $[g, h] = e \implies (gh)^\alpha = g^\alpha h^\alpha$ ($MR$-axiom).

Definition 1 ([1]). The group $G$ is called an exponential $R$-group (or $R$-group) after Lyndon if an action of the ring $R$ on $G$ satisfying axioms (i)–(iii) is given.

Definition 2 ([2]). The group $G$ is called an exponential $R$-group (or $MR$-group) if an action of the ring $R$ on $G$ satisfies axioms (i)–(iv). Then $R$ is called a ring of scalars of the group $G$.
Let \( \mathcal{L}_R \) and \( \mathcal{M}_R \) be the classes of all exponential \( R \)-group after Lyndon and all \( MR \)-groups, \( \mathcal{L}_R \supseteq \mathcal{M}_R \). Any Abelian \( MR \)-group is an \( R \)-module, and vice versa. There exist Abelian Lyndon \( R \)-groups that are not \( R \)-modules (see [10], where the structure of free abelian \( R \)-group is extensively investigated), i.e. \( \mathcal{L}_R \supset \mathcal{M}_R \).

Most of natural examples of exponential \( R \)-groups lie in \( \mathcal{M}_R \). For example, unipotent groups over a field \( K \) of zero characteristic are \( MR \)-groups, pro-\( p \)-groups are exponential \( M\mathbb{Z}_p \)-groups over the ring \( \mathbb{Z}_p \) of \( p \)-adic integers, etc (see [2] for examples).

**Definition 3 ([2])**. A homomorphism of \( R \)-groups \( \varphi : G \to H \) will be called an \( R \)-homomorphism if \( \varphi(g^\alpha) = \varphi(g)^\alpha \), \( g \in G, \alpha \in R \).

For basic definitions in category \( \mathcal{M}_R \) and results concerning exponential \( MR \)-group see [2].

Let \( R \) be an arbitrary associative ring unit. Then the class \( \mathcal{M}_R (\mathcal{L}_R) \) is a category in which morphisms are \( R \)-homomorphism of groups.

Below we prove that the classes \( \mathcal{L}_R \) and \( \mathcal{M}_R \) are closed un under direct and Cartesian products and under direct and inverse limits.

Let \( G_i \in \mathcal{L}_R, i \in I \). We denote by \( \prod G_i \) and \( \prod G_i \), respectively, the Cartesian and direct products of the groups \( G_i \). Let \( g \in \prod G_i, g = (\ldots, g_i, \ldots) \), and \( \alpha \in R \). We define an action of \( R \) on \( G \) by the coordinate-wise rule \( g^\alpha = (\ldots, g_i^\alpha, \ldots) \). It can be immediately proved that if all groups \( G_i \) satisfy one of axioms (i)–(iv), then the groups \( \prod G_i \) and \( \prod G_i \), also satisfy this axiom. Thus, we have proved

**Proposition 1.** The classes \( \mathcal{L}_R \) and \( \mathcal{M}_R \) are closed with respect to direct and Cartesian product.

If in the standard definitions of direct and inverse spectrums one considers only \( R \)-homomorphisms then it is not difficult to prove

**Proposition 2.** The classes \( \mathcal{L}_R \) and \( \mathcal{M}_R \) are closed with respect to direct and inverse limits.

It is proved in [11] that in Abelian group category the operations of direct product of groups, of direct and inverse limits have a universal property. The corresponding actions in exponential \( MR \)-group category have analogous properties.

**Definition 4 ([2])**. Let \( G \) be an \( MR \)-group and let \( \mu : R \to S \) be a homomorphism of rings. Then a \( S \)-group \( G^{S,\mu} \) is called the tensor \( S \)-completion of an \( MR \)-group \( G \), if \( G^{S,\mu} \) satisfies the following universal property:

1. there exists and \( R \)-homomorphism \( \lambda : G \to G^{S,\mu} \) such that \( \lambda(G) \) \( S \)-generators \( G_{s,\mu} \), i.e. \( \langle \lambda(G) \rangle = G^{S,\mu} \);
2. for any \( S \)-group \( H \) and any \( R \)-homomorphism \( \varphi : G \to H \) coordinated with \( \mu \) (that \( \varphi(g^\alpha) = \varphi(g)^{\mu(\alpha)} \)) there exists a \( S \)-homomorphism \( \psi : G^{S,\mu} \to H \), rendering the
following diagram commutative:

\[
\begin{array}{ccc}
G & \xrightarrow{\lambda} & G^{S,\mu} \\
\varphi \downarrow & & \downarrow \exists \psi \\
H & \xrightarrow{\phi = \lambda \psi} & G^S 
\end{array}
\]

Note that if \( G \) is an Abelian MR-group, then \( G^{S,\mu} \cong G \otimes_S R \) is a tensor product of an \( R \)-module \( G \) by a ring \( S \). In [2] it is proved that for any MR-group \( G \) and any homomorphism \( \mu : R \to S \) the tensor completion \( G^{S,\mu} \) always exists and it is unique to written an isomorphism.

2 Commutation of the functor of tensor completion with basic group operations. Let \( G_i \in \mathfrak{M}_R, i \in I \).

Theorem 1. If \( G = \prod_i G_i \), then \( G^S = \prod_i G^S_i \).

Theorem 2. If \( G_* = \lim_i G_i \), then \( G_*^S = \lim_i G_i^S \).

Remark 1. Let us give an example showing that the operation of Cartesian product is not commutable with the operation of tensor completion. Denote \( \lambda : \prod_i G_i \to \prod_i G_i^S \).

Then by the universal property of tensor completion, we have a \( S \)-homomorphism \( \lambda^* : (\prod_i G_i)^S \to \prod_i G_i^S \), which is not an isomorphism in the general case. Such an example exists in the theory of Abelian groups. Let us take the field of rational numbers \( \mathbb{Q} \) as the ring \( R \) a cyclic group of order \( n \) as \( G_n \). Let \( G_n = \langle a_n \rangle, n \in \mathbb{N} \). Then \( G_n^\mathbb{Q} = G_n \otimes \mathbb{Q} = 0 \).

Therefore, \( \prod_n G_n^\mathbb{Q} = 0 \). At the same time there exist elements of infinite order in the group \( \prod_n G_n \), therefore the group \( (\prod_n G_n)^\mathbb{Q} = \prod_n G_n \otimes \mathbb{Q} \) is nonzero.

Remark 2. Let \( G_* \) be the limits group of inverse spectrum \( \mathbb{G}^* = \{ G_i, i \in I, \pi^i \} \).

By the use of universal property of inverse limit we shall construct a \( S \)-homomorphism \( \sigma : (G^*)^S \to \lim_i G_i^S \). To do this we denote by \( \pi_i : G^* \to G_i \) the projection of the limit group onto the component with index \( i \). Then \( \pi_i^* : (G^*)^S \to G_i^S \) is the corresponding homomorphism of a tensor completion. Let \( \mu_i : \lim_i G_i^S \to G_i^S \) be a natural project.

Then according to universal property of inverse limits, there exists a homomorphism \( \sigma : (G^*)^S \to \lim_i G_i^S \), making the diagram

\[
\begin{array}{ccc}
(G^*)^S & \xrightarrow{\sigma} & \lim_i G_i^S \\
\downarrow \pi_i^* & & \downarrow \mu_i \\
G_i^S & &
\end{array}
\]
We shall demonstrate with an example that this homomorphism $\sigma$ is not isomorphism in the general case. Let us consider $G_n$, $n \in \mathbb{N}$, $G_n = \langle a_n \rangle$, where $a_n$ is an element of order $p^n$, $p$ is a prime number. Then it is known that $\lim_{n \to \infty} G_n \cong \mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty}$ is the additive group of integral $p$-adic numbers, $\mathbb{Z}_p^\infty = \mathbb{Z}_{p^\infty} \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$ of continual cardinality. At the same time $\lim_{n \to \infty} G_n^\mathbb{Q} = \lim_{n \to \infty} (G_n \otimes \mathbb{Q}) = \lim_{n \to \infty} 0$.

REFERENCES