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THE GREEN FUNCTION FOR TIME INDEPENDENT MULTI-VELOCITY TRANSPORT EQUATION

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Abstract. The aim of this paper is to construct the Greens function for the one simple time independent multi-velocity transport equation, describing neutron diffusion in a uniform infinite medium. To this end we used the method of expansions by the elementary regular and singular eigenfunctions of the corresponding characteristic equation.

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Consider the following simple transport equation

$$\mu\frac{\partial\psi}{\partial x}+\psi=\int_{E_1}^{E_2}\int_{-1}^{+1}\psi(x,\mu',E')d\mu'dE' \tag{1}$$

$$x\in(-\infty,+\infty),\quad \mu\in(-1,+1),\quad E\in[E_1,E_2]$$

Translational invariance suggests trying

$$\psi(x,\mu,E)=\exp^{-x/\nu}\phi_{\mu,E}.$$

With this assumption, Eq(1) becomes

$$(1-\frac{\mu}{\nu})\phi_\nu(\mu,E)=\int_{E_1}^{E_2}\int_{-1}^{+1}\phi(\mu',E')d\mu'dE' \tag{2}$$

It is very convenient to normalize so that

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\phi(\mu',E')d\mu'dE'=1 \tag{3}$$

With this assumption, Eq(1) becomes

$$(\nu-\mu)\phi_\nu(\mu,E)=\nu \tag{4}$$

From this point the conventional argument runs as follows: Solving Eq.(4)gives

$$\phi_\nu(\mu,E)=\frac{\nu}{\nu-\mu} \tag{5}$$

Inserting this result into Eq.(3)yields the condition

$$\Lambda(\nu) \equiv 1 - \frac{c\nu}{2} ln \frac{1 + 1/\nu}{1 - 1/\nu} = 0,$$

(here $c = 2(E_2 - E_1)$). There are, is well known [1] two roots of the equation. Here they will be denoted by $\pm\nu_0$. (We note that for $c < 1$, $Re 1/\nu_0 < 1$ and ν_0 is purely imaginary for $c > 1$). The argument has given the usual solutions of the homogeneous transport equation. However, there are others. Thus, Eq.(5)does not strictly follow from Eq.(4) if $\phi_\nu(\mu, E)$ can be a distribution (see[2]). Thus from Eq.(4) one can only conclude that

$$\phi_{\nu,(\zeta)}(\mu, E) = \frac{\nu}{\nu - \mu} + \left(\delta(\zeta - E) - \int_{-1}^{+1} \frac{\nu}{\nu - \mu'} d\mu' \right) \delta(\nu - \mu). \tag{6}$$

There are possibilities: There two roots $\pm\nu_0$ occur. With the normalization (3) the corresponding solutions of the transport equation are

$$\psi_{0\pm}(x, \mu, E) = \phi_{0\pm}(\mu, E) e^{\mp/\nu_0} \tag{7}$$

where

$$\phi_{0\pm}(\mu, E) = \frac{\nu_0}{\nu_0 \mp \mu}.$$

Also, there are singular solutions for all real ν such that $-1 \leq \nu \leq 1$

$$\psi_{\nu,(\zeta)}(x, \mu, E) = \phi_{\nu,(\zeta)}(\mu, E) e^{-x/\nu}. \tag{8}$$

To summarize: There are two discrete solutions given by Eq.(7) and a continuum of solutions given by Eq.(8).

The usefulness of these functions arises from the facts that they are both orthogonal and complete. This can be stated in the form of two theorems (see[2]). Let

$$\tilde{\phi}_{\nu,(\zeta)}(\mu, E) = \phi_{\nu,(\zeta)}(\mu, E) + \frac{g(\nu)}{1 - \frac{c}{2} g(\nu)} \int_{E_1}^{E_2} \phi_{\nu,(\zeta')}(\mu, E) d\zeta'$$

where

$$g(\nu) = -\pi^2 \nu^2 \frac{c}{2} + 2 \int_{-1}^{+1} \frac{\nu}{\nu - \mu} d\mu - \frac{c}{2} \left(\int_{-1}^{+1} \frac{\nu}{\nu - \mu} d\mu \right)^2$$

Theorem 1.

(a)

$$\int_{E_1}^{E_2} \int_{-1}^{+1} \mu \phi_\nu(\mu, E) \phi_{\nu'}(\mu, E) d\mu dE = 0, \quad \nu \neq \nu'$$

(b)

$$\int_{E_1}^{E_2} \int_{-1}^{+1} \mu \phi_{\nu,(\zeta)}(\mu, E) \tilde{\phi}_{\nu',(\zeta')}(\mu, E) d\mu dE = \delta(\nu - \nu') \delta(\zeta - \zeta')$$

(c)

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{0\pm}(\mu,E)\phi_{\nu,(\zeta)}(\mu,E)d\mu dE=0$$

Theorem 2.

The functions $\phi_{0\pm}(\mu,E)$ and $\phi_{\nu,(\zeta)}(\mu,E)$ $\nu\in(-1,+1)$, $\zeta\in[E_1,E_2]$ are complete for the functions $\psi(\mu,E)$ defined in the $-1\le\mu\le1$; $E_1\le E\le E_2$. i.e. one can express $\psi(\mu,E)$ in the form

$$\psi(\mu,E)=a_{0+}\phi_{\mu,E}+a_{0-}\phi_{0-}(\mu,E)+\int_{E_1}^{E_2}\int_{-1}^{+1}u(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)d\nu d\zeta.$$

It follows from Theorem 1 that

$$a_{0\pm}=\frac{1}{N_{0\pm}}\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{0\pm}(\mu,E)\psi(\mu,E)d\mu dE,$$

$$N_{0\pm}=\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{0\pm}^2(\mu,E)d\mu dE$$

and

$$u(\nu,\zeta)=\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\tilde{\phi}_{\nu,(\zeta)}(\mu,E)\psi(\mu,E)d\mu dE.$$

As an illustration of the applicability of the results the Green function for the transport equation can be constructed. To be definite $c<1$ is assumed here. The Green function ψ_g satisfies the equation

$$\mu\frac{\partial\psi_g}{\partial x}+\psi_g=\int_{E_1}^{E_2}\int_{-1}^{+1}\psi_g(x,\mu',E')d\mu'dE'+\frac{\delta(x)}{4\pi}\delta(\mu-\mu_0)\delta(E-E_0),$$

$$x\in(-\infty,+\infty),\quad \mu\in(-1,+1),\quad E\in[E_1,E_2]$$

and the jump condition

$$\mu(\psi_g(0^+,\mu,E)-\psi_g(0^-,\mu,E))=\frac{1}{4\pi}\delta(\mu-\mu_0)\delta(E-E_0).$$

R E F E R E N C E S

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