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SPECIAL EXP-FUNCTION METHOD FOR TRAVELING WAVE SOLUTIONS OF BURGER'S EQUATION

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Abstract. Using the special exp-function method traveling wave exact solutions of 2D nonlinear Burger's equation are obtained. It is shown that such solutions have spatially isolated structural (soliton-like) forms. Revision of the previously received solutions is carried out.

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1 Introduction. Nonlinear Burger's equation describes shock waves in liquid and gas. It can be also used to model vehicles density on motor roads. Burger's equation connects the dissipative uu_x term with the convectional uu_x one (see below Eq. (10)). Using the straightforward method used by Tsamalashvili in [1] soliton like exact solutions are obtained for the 2D nonlinear modified Burger's equation. Employing the special exp-function expansion method Mohyud-Din et al. [2] constructed exact traveling wave solutions for (2+1) - dimensional Burger's equation. Unfortunately this paper contains numerous wrong results and our main purpose is to revise previously obtained solutions.

2 Special exp-function expansion method. Consider the following nonlinear partial differential equation for $u(x,t)$:

$$\Phi\left(u, u_t, u_x, u_{xx}, u_{tt}, u_{xxx}, \ldots\right) = 0, \tag{1}$$

where Φ is a polynomial of u and its derivatives. To find the traveling wave solution we introduce the new variable $\eta = x - Vt$, where V is the speed of wave. Then Eq. (1) can be reduced to the following ODE for $u = u(\eta)$

$$G\left(u, u', u'', u''', \ldots\right) = 0. \tag{2}$$

The use of special exp-function expansion method implies the possibility to find the exact solution of a nonlinear evolutionary equation by the function $exp\left(-\varphi(\eta)\right)$, where the $\varphi(\eta)$ function satisfies the ODE.

$$\frac{d\varphi(\eta)}{d\eta} = e^{-\varphi(\eta)} + \mu e^{\varphi(\eta)} + \lambda\ . \tag{3}$$

Here μ and λ are parameters. Thus we are seeking the solution of Eq. (2) by the following finite series

$$u(\eta) = \sum_{n=0}^{M} a_n \exp\left(-n\varphi(\eta)\right), \tag{4}$$

where a_n and $0 \leq n \leq M$ are constant, $a_n \neq 0$ and the function $\varphi(\eta)$ satisfies Eq. (3). It is clear that the solution of Eq. (1) depends on the relation between μ and λ. Namely

1) when $\lambda^2 - 4\mu > 0$, $\mu \neq 0$, we have

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left[-\sqrt{\lambda^2 - 4\mu}\tan h\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,(\eta + c_1)\right) - \lambda\right]\right\}. \tag{5}$$

2) when $\lambda^2 - 4\mu < 0$, $\mu \neq 0$, then

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left[\sqrt{4\mu - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\,(\eta + c_1)\right) - \lambda\right]\right\}. \tag{6}$$

3) when $\lambda^2 - 4\mu > 0$, $\mu = 0$, $\lambda \neq 0$, then

$$\varphi(\eta) = -\ln\left\{\frac{\lambda}{\exp\left[\lambda\,(\eta + c_1)\right] - 1}\right\}. \tag{7}$$

4) when $\lambda^2 - 4\mu = 0$, $\lambda \neq 0$, $\mu \neq 0$, we have

$$\varphi(\eta) = \ln\left\{-\frac{2\left[\lambda\,(\eta + c_1) + 2\right]}{\lambda^2\,(\eta + c_1)}\right\}. \tag{8}$$

5) when $\lambda^2 - 4\mu = 0$, $\lambda = 0$, $\mu = 0$, we have

$$\varphi(\eta) = \ln\left(\eta + c_1\right). \tag{9}$$

Note that Eqs: (7) and (9) of [2] are wrong by use of wrong signs. By considering the homogenous principle in Eq. (2) we examine the last $\exp\left(-M(\varphi(\eta)\right)$ term in the expansion (4) and substitute in Eq. (2). We get algebraic equations with a_M, V, λ, μ, after comparing the same powers of $\exp\left(-\varphi(\eta)\right)$ to zero. We substitute such solutions in (4) and by use of expressions (5)-(9) we get some valuable traveling wave solutions of Eq. (1).

3 Solution of (2+1)-dimensional Burger's equation. Consider the nonlinear Burger's equation

$$u_t - uu_x - u_{xx} - u_{yy} = 0. \tag{10}$$

Here subscripts indicate the appropriate derivatives over time t and space x, y. After the passage to the traveling waves argument $\eta = x + y - Vt$ we get the following ODE for $u = u(\eta)$ function

$$-Vu' - 2u'' - uu' = 0, \tag{11}$$

where prime indicates the derivative over the variable η. Integration of Eq. (11) gives

$$-Vu - 2u' - \frac{1}{2}u^2 + c = 0. \tag{12}$$

Using the homogenous principle, balancing u' and u^2, we have $M = 1$. Then the trial solution of Eq. (11) can be stated as

$$u(\eta) = a_1 \exp\left(-\varphi(\eta)\right) + a_0, \tag{13}$$

where $a_1 \neq 0$, a_0 is a constant. By putting u, u' and u^2 in Eq. (12) and comparing, we get the system

$$\begin{cases} -\dfrac{1}{2}a_0^2 + 2a_1\mu + c - Va_0 = 0, \\ -a_0a_1 + 2a_1\lambda - Va_1 = 0, \\ -\dfrac{1}{2}a_1^2 + 2a_1 = 0. \end{cases} \tag{14}$$

By solving the algebraic system, the required solution is (differs from the appropriate wrong expression of [2])

$$\lambda = \pm\frac{1}{2}\sqrt{V^2 + 2c + 16\mu}, \quad a_0 = -V \pm \sqrt{V^2 + 2c + 16\mu}, \quad a_1 = 4, \tag{15}$$

where λ and μ are constants. Now putting these obtained solutions in Eq. (13), we get the solution

$$u = -V \pm \sqrt{V^2 + 2c + 16\mu} + 4e^{-\varphi(\eta)}, \tag{16}$$

where $\eta = x + y - Vt$.

Now putting (5)-(9), we obtain the following solutions (differ from the appropriate wrong solutions of [2]):

1) When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$, we have

$$u_1(\eta) = -V + 2\lambda - \frac{8\mu}{\sqrt{\lambda^2 - 4\mu}\tan h\left[\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}(\eta + c_1)\right] + \lambda}, \tag{17}$$

where $\eta = x + y - Vt$ and c_1 is an arbitrary constant.

2) When $\lambda^2 - 4\mu < 0$, and $\mu \neq 0$, we get

$$u_2(\eta) = -V + 2\lambda + \frac{8\mu}{\sqrt{-\lambda^2 + 4\mu}\tan\left[\dfrac{\sqrt{-\lambda^2 + 4\mu}}{2}(\eta + c_1)\right] - \lambda}, \tag{18}$$

where $\eta = x + y - Vt$ and c_1 is an arbitrary constant.

3) When $\lambda^2 - 4\mu > 0$, and $\mu = 0$, $\lambda \neq 0$, we have

$$u_3(\eta) = 2\lambda - V + 4\lambda\left[e^{\lambda(\eta + c_1)} - 1\right]^{-1}. \tag{19}$$

Here $a_0 = 2\lambda - V$, $a_1 = 4$, $\lambda = \pm\sqrt{\dfrac{V^2}{4} + \dfrac{c}{2}}$, and $\eta = x + y - Vt$.

4) When $\lambda^2 - 4\mu = 0$, $\lambda \neq 0$, $\mu \neq 0$, we obtain

$$u_4(\eta) = 2\lambda - V - 2\lambda^2 \frac{\eta + c_1}{2 + \lambda(\eta + c_1)}. \tag{20}$$

Here $a_0 = 2\lambda - V$, $a_1 = 4$, $c = -V^2/2$ and $\eta = x + y - Vt$.

5) When $\lambda = 0$, $\mu = 0$, we get

$$u_5(\eta) = \frac{4}{\eta + c_1} - V. \tag{21}$$

Here $a_0 = -V$, $a_1 = 4$, $c = -V^2/2$ and $\eta = x + y - Vt$.

4 Summary. The special exp-function method can be successfully applied to find exact traveling wave solutions of the nonlinear (2+1)-dimensional Burger's equation. The found solutions represent exact solitary wave solutions of soliton and kink soliton forms which are expressed through the hyperbolic, trigonometric and rational functions. Revision of the previously received solutions is carried out.

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