Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 30, 2016

ON THE SOLUTION OF THE NON-CLASSICAL PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN

Kosta Svanadze

Abstract. In the paper for the homogeneous equation of statics of the linear theory of elastic mixture in a circle one non-classical problem is solved, which for the case of a circular domain is analogous to non-classical problems for harmonic and biharmonic equations considered by A. Bitsadze. The solution is represented by absolutely and uniformly convegent series.

Keywords and phrases: Elastic mixture, Kolosov-Muskhelishvili type formulas, Fourier series, non-classical problem.

AMS subject classification (2010): 74B05.

1 Introduction. The basic two-dimesional boundary value problems statics of the linear theory of elastic mixtures are studied in [1], [2], [5] and also by many other authors.

In the monograph [3] A. Bitsadze considered the non-classical boundary value problem in a circle for a harmonic equation in case of the Dirichlet problem. In this work we consider the non-classical boundary value problem in the circle for the homogeneous equation of statics of the linear theory of elastic mixture.

2 Basic equation and boundary value problem. The homogeneous equation of static of the linear theory of elastic mixture compex form is written as [2]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z^2}} = 0, \qquad U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1}$$

where $u_p, \quad p = \overline{1,4}$, are components of the displacement vector, $z = x_1 + ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),$$

$$K = -\frac{1}{2} l m^{-1}, \quad l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1}, \quad det\ |m| > 0,$$

$m_k, \quad l_{3+k}, \quad k = 1, 2, 3$ are expressed in terms of elastic constants [2] or [5].

The following formulas are valid [1]

$$\frac{2(m_1 + X_j m_2)}{l_4 + X_j l_5} = \frac{2(m_2 + X_j m_3)}{l_5 + X_j l_6} = -\frac{1}{K_j}, \quad K_j \neq 0, \quad |K_j| < 1, \quad j = 1, 2, \tag{2}$$

where $X_j (j = 1, 2)$ is a real constant, $X_1 - X_2 \neq 0$.

In [2] M. Basheleischvili obtained the following representations (Kolosov-Muskhelishvili type formulas)

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2} lz\overline{\varphi')z)} + \overline{\psi(z)},$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)} \left(-2\varphi(z) + 2\mu U(x)\right), \tag{3}$$

where $\varphi = (\varphi_1 \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions; $(TU)_p$, $p = \overline{1, 4}$ are components of the stress vector [5],

$$\mu = \begin{bmatrix} \mu_1 \mu_3 \\ \mu_3 \mu_2 \end{bmatrix}, \qquad m = \begin{bmatrix} m_1 m_2 \\ m_2 m_3 \end{bmatrix}, \qquad det|\mu| > 0,$$

$\frac{\partial}{\partial S(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$, $\quad n = (n_1, n_2)^T$ is the unit vector of the outer normal.

The following Green formula is valid [5]

$$\int_D E(u, u) dx = I_m \int_S U \overline{TU} ds, \tag{4}$$

where D is a circular domain $D = (|z| < 1)$ and S is a boundary of D, $E(u, u)$ is the positively defined quadratic form. The equation $E(u, u) = 0$ admits the solution

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = \nu + i\varepsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad z = x_1 + ix_2, \tag{5}$$

where $\nu = (\nu_1, \nu_2)^T$ is an arbitrary complex vector and ε is an arbitrary real constant.

Let us formulate the non-classical boundary value problem in the circular domain D. Find a vector $U = (u_1 + iu_2, u_3 + iu_4)^T$ which belongs to the class $C^2(D) \bigcap C^{1,\alpha}(D \bigcup S)$, is a solution of equation (1) and satisfies the following boundary condition

$$U(e^{i\theta}) - U(\delta e^{i\theta}) = f(e^{i\theta}), \quad 0 \leq \theta \leq 2\pi, \quad 0 < \delta < 1, \tag{6}$$

where $f = (f_1, f_2)^T$ is a given vector-function, below we assume that f'' belongs to the Dirichlet class.

Using the Green formula (4) it is easy to prove (see(5)).

Theorem. *Two arbitrary regular solutions of problem (1),(6) differ from each other only by a constant vector.*

On the basis of formula $(3)_1$ *our problem (1), (6) is reduced to finding two analytic vector-functions* $\varphi(z)$ *and* $\psi(z)$ *in* D *by the boundary condition*

$$m\varphi(t) + \frac{1}{2} lt\overline{\varphi'(t)} + \overline{\psi(t)} - m\varphi(\delta t) - \frac{1}{2} l \;\; \delta t\overline{\varphi'(\delta t)} - \overline{\psi(\delta t)} = f(t),$$

$$t = e^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad 0 < \delta < 1. \tag{7}$$

Let us expand the boundary vector-function $f(e^{i\theta})$ into the Fourier series

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \quad 0 \le \theta \le 2\pi, \tag{8}$$

where $A_n = (A_{n_1}, A_{n_2})^T$ is a known coefficient.

We look for the vector-function $\varphi(z)$ and $\psi(z)$ in the following form [4]

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k, \quad \psi(z) = \sum_{k=0}^{\infty} b_k z^k, \tag{9}$$

where $a_k = (a_{k_1}, a_{k_2})^T$ and $b_k = (b_{k_1}, b_{k_2})^T$ are unknown coefficients.

Taking into account (8) and (9) in (7) we obtain the following system of equations for unknown coefficitns:

$$ma_1 + \frac{1}{2} l\overline{a_1} = \frac{1}{1-\delta} A_1, \tag{10}$$

$$m(1-\delta^n)a_n = A_n, \quad (n > 1), \tag{11}$$

$$\frac{1}{2} l(1-\delta^{n+2})\overline{a_{n+2}} + (1-\delta^n)\overline{b_n} = A_{-n}, \quad (n \ge 0). \tag{12}$$

It is evindent that from the systems (11) and (12) the coefficitns $a_n(n > 1)$ and $b_n(n \ge 0)$ are defined uniquely.

Using (2) we determine the coefficient a_1. The system (10) can be rewritten as follows

$$(a_1 - K_j\overline{a_1}) \begin{pmatrix} m_1 + X_j m_2 \\ m_2 + X_j m_3 \end{pmatrix} = \frac{A_1}{1-\delta} \begin{pmatrix} 1 \\ X_j \end{pmatrix}, \quad j = 1, 2, \tag{13}$$

whence we have

$$(\overline{a_1} - K_j a_1) \begin{pmatrix} m_1 + X_j m_2 \\ m_2 + X_j m_3 \end{pmatrix} = \frac{\overline{A_1}}{1-\delta} \begin{pmatrix} 1 \\ X_j \end{pmatrix}, \quad j = 1, 2. \tag{14}$$

from (13) and (14) we'll have

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \begin{pmatrix} m_1 + X_j m_2 \\ m_2 + X_j m_3 \end{pmatrix} = \frac{1}{(1-\delta)(1-K_j^2)} (A_1 + K_j\overline{A_1}) \begin{pmatrix} 1 \\ X_j \end{pmatrix}, \quad j = 1, 2. \tag{15}$$

Remarking that determinant of the system (15) is equal to $(X_2 - X_1)det|m|$ and different from zero. Consequently, the coefficient $a_1 = (a_{11}, a_{12})^T$ is defined uniquely.

By the above given reasoning we have the coefficients $a_n(n \ge 1)$ and $b_n(n \ge 0)$ are defined uniquely by means of Fourier coefficients of the $f(e^{i\theta})$. Further, note that f'' belong to the Dirichlet class (see 2^0).

Having found the coefficients $a_n (n \geq 1)$ and $b_n (n \geq 0)$ using formulas (9) we can find $\varphi(z)$ and $\psi(z)$ given by absolutely and uniformly convergent series (see [4]).

Using the expressions of the above-mentioned vector-functions and substituting them into the expression for the displacement vector (see $(3)_1$ we obtain the solution of the posed problem given by absolutely and uniformly convergent series.

REFERENCES

1. Basheleishvili, M. Application of analogues of general Kolosov-Muskhelishvili representations in the theory of elastic mixtures. *Georgian Math. J.*, **6**, 1 (1999), 1-18.

2. Basheleishvili, M., Svanadze K. A new method of solving the basic plane boundary value problems of statics of the elastic mixture theory. *Georgian Math. J.*, **8**, 3 (2001), 427-446.

3. Bitsadze, A. Some Classes of Partial Differential Equations (Russian). *Moscow*, 1981.

4. Muskhelishvili, N. Some Basic Problems of the Mathematical Theory of Elasticity (Russian). *Moscow*, 1966.

5. Svanadze, K. Solution of the basic plane boundary value problems of statics of the elastic mixture for a multiply connected domain by the method of D. Sherman. *Semin. I. Vekua Inst. Appl. Math. Rep.*, **39** (2013), 58-71.

Received 30.05.2016; revised 30.11.2016; accepted 28.12.2016.

Author(s) address(es):

Kosta Svanadze
A. Tsereteli Kutaisi State University
Tamar Mepe str. 59, 4600 Kutaisi, Georgia
E-mail: kostasvanadze@yahoo