Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 30, 2016

ON THE SPACE OF SPHERICAL POLYNOMIAL WITH QUADRATIC FORMS OF ANY NUMBER OF VARIABLES

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Abstract. The spherical polynomials of order ν with respect to the quadratic form of r variables are constructed and the basis of the space of these spherical polynomials is established. The space of generalized theta-series with respect to the quadratic form of r variables is considered.

Keywords and phrases: Quadratic form, spherical polynomial, generalized theta-series.

AMS subject classification (2010): 11E20, 11E25, 11F27.

1 Introduction.

Let

$$Q(X) = Q(x_1, \cdots, x_r) = \sum_{1\leq i\leq j\leq r} b_{ij}x_ix_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. Let A_{ij} denote the cofactor to the element a_{ij} in A and a^*_{ij} is the element of the inverse matrix A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\leq i,j\leq r} a^*_{ij}\Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0 \tag{1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n)z^{Q(n)}, \qquad z = e^{2\pi i\tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0$$

is the corresponding generalized r-fold theta-series.

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$, $\dim P(\nu, Q) = \binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$. For $\nu = 4$, Lomadze [3] constructs the basis of the space of spherical polynomials of fourth order with respect to $Q(X)$.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \,:\, P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic forms Q. Gaigalas [4] gets the upper bounds for the dimension of the space $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms. In [5-7] we established the upper bounds for the dimension of the space $T(\nu, Q)$ for some quadratic forms of r variables, when $r = 3, 4, 5$, in a number of cases we calculated the dimension and form the bases of these spaces.

In this paper we form the basis of the space of spherical polynomials of order ν with respect to $Q(X)$ of r variables and obtained the upper bounds for the dimension of the space $T(\nu, Q)$ for some diagonal quadratic forms of any number of variables.

2 On the basis of the space $P(\nu, Q)$ and $T(\nu, Q)$. Let

$$P(X) = P(x_1, x_2, x_3, \cdots x_r) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \cdots \sum_{l=0}^{m} a_{kij\cdots l} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} \cdots x_r^{l}$$

be a spherical polynomial of order ν with respect to the positive quadratic form $Q(x_1, x_2, \cdots, x_r)$ of r variables and

$$L = [a_{000\cdots 0}, a_{100\cdots 0}, a_{110\cdots 0}, a_{111\cdots 0}, \ldots, a_{\nu\nu\nu\cdots\nu}]^T$$

be the column vector, where $a_{kij\cdots l}$ $(\nu \geq k \geq i \geq j \geq \cdots \geq l \geq 0)$ are the coefficients of polynomial $P(X)$.

Conditions (1) in matrix notation have the following form $S \cdot L = 0$, where the matrix S (the elements of this matrix are defined from conditions (1)) has the form

$$\begin{Vmatrix} A_{11}(\nu-1)\nu & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & 2A_{15}(\nu-1) & \ldots & 0 \\ 0 & A_{11}(\nu-1)\nu & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & A_{11}(\nu-1)\nu & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & A_{11}(\nu-1)\nu & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & A_{rr}(\nu-1)\nu \end{Vmatrix}.$$

The number of rows of the matrix S is equal to

$$\sum_{k=1}^{\nu-1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} \cdots \sum_{m=0}^{s} \sum_{l=0}^{m} 1 = \sum_{k=1}^{\nu-1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} \cdots \sum_{m=0}^{s} (m+1) = \binom{\nu+r-3}{r-1}$$

and the number of columns of the matrix S is equal to

$$\sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \cdots \sum_{m=0}^{s} \sum_{l=0}^{m} 1 = \binom{\nu+r-1}{r-1}.$$

Hence S is $\binom{\nu+r-3}{r-1} \times \binom{\nu+r-1}{r-1}$ matrix.

We partition the matrix S into two matrices S_1 and S_2. S_1 is the left square nondegenerate $\binom{\nu+r-3}{r-1} \times \binom{\nu+r-3}{r-1}$ matrix, it consists of the first $\binom{\nu+r-3}{r-1}$ columns of the matrix S; matrix S_2 consists of the last $\binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ columns of the matrix S.

Similarly, we partition the matrix L into two matrices L_1 and L_2. L_1 is the $\binom{\nu+r-3}{r-1}\times 1$ matrix, it consists of the upper $\binom{\nu+r-3}{r-1}$ elements of the matrix L; L_2 consists of the lower $\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ elements of the matrix L.

According to the new notation, the matrix equality has the form $S_1L_1+S_2L_2=0$, i.e., $L_1=-S_1^{-1}S_2L_2$.

It follows from this equality that the matrix L_1 is expressed through the matrix L_2, i.e., the first $\binom{\nu+r-3}{r-1}$ elements of the matrix L are expressed through its other elements. Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu+r-3}{r-1}$ coefficients can be expressed through the last $\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ coefficients.

Let $Q(X)=Q(x_1,x_2,x_3,x_4,\dots,x_r)$ be a quadratic form of r variables. We have, $\dim\mathcal{P}(\nu,Q)=\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ and we have proved the following

Theorem 1. *The polynomials (the coefficients of polynomial $P_{abc\dots d}$ are given in the brackets, where $abc\dots d$ are the indices of the coefficient equaled to one from $a_{\nu-1,00\dots0}$ to $a_{\nu,\nu,\nu,\dots,\nu}$)*

$$P_{\nu-1,00\dots0}(a^{(1)}_{000\dots0},a^{(1)}_{100\dots0},\dots,a^{(1)}_{\nu-2,\nu-2,\nu-2,\dots,\nu-2},1,0,0,\dots,0),$$

$$P_{\nu-1,10\dots0}(a^{(2)}_{000\dots0},a^{(2)}_{100\dots0},\dots,a^{(2)}_{\nu-2,\nu-2,\nu-2,\nu-2},0,1,0,\dots,0),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$P_{\nu,\nu,\nu\dots\nu}(a^{(t)}_{000\dots0},a^{(t)}_{100\dots0},\dots,a^{(t)}_{\nu-2,\nu-2,\nu-2,\dots,\nu-2},0,0,0,\dots,1),$$

where the first $\binom{\nu+r-3}{r-1}$ coefficients from $a_{000\dots0}$ to $a_{\nu-2,\nu-2,\nu-2,\dots,\nu-2}$ are calculated through other $\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ coefficients, form the basis of the space $\mathcal{P}(\nu,Q)$.

Consider the generalized r-fold theta-series

$$\vartheta(\tau,P,Q)=\sum_{n\in\mathbb{Z}^r}P(n)z^{Q(n)},\qquad z=e^{2\pi i\tau}.$$

Our goal is to construct a basis of the space of generalized theta-series with spherical polynomial P of order ν for diagonal quadratic form Q of r variables.

Construct the integral automorphisms U of the diagonal quadratic form

$$Q(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+\dots+b_{rr}x_r^2.$$

An integral $r\times r$ matrix U is called an integral automorphism of the quadratic form $Q(X)$ in r variables if the condition $U^TAU=A$ is satisfied.

The integral automorphisms of the quadratic form $Q(X)$ are

$$U=\begin{Vmatrix} e_1 & 0 & 0 & \dots & 0\\ 0 & e_2 & 0 & \dots & 0\\ 0 & 0 & e_3 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & e_r\end{Vmatrix},\qquad \text{where}\qquad e_i=\pm1.$$

It is known ([1], p. 37) that, if G is the set of all integral automorphisms of Q and

$$\sum_{i=1}^{t} P(U_i X) = 0 \qquad \text{for some} \qquad U_1, \ldots, U_t \in G, \qquad \text{then} \qquad \vartheta(\tau, P, Q) = 0.$$

Consider all possible sums $\sum_{i=1}^{t} P(U_i X) = 0$. For such polynomials $\vartheta(\tau, P, Q) = 0$. If among the last $\binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ coefficients of P, at least one of indices $k, i, j, \ldots l$ of the coefficient, equaled to one, is odd, then spherical polynomials $P = P_{kij\ldots l}$ satisfies the equality $\vartheta(\tau, P, Q) = 0$. Hence the maximal number of linearly independent theta-series (when the indices $k, i, j, \ldots l$ of the corresponding spherical polynomial P is even) is

$$\sum_{\substack{i=0\\2|i}}^{\nu} \sum_{\substack{j=0\\2|j}}^{i} \cdots \sum_{\substack{m=0\\2|m}}^{s} \sum_{\substack{l=0\\2|l}}^{m} 1 = \sum_{\substack{i=0\\2|i}}^{\nu} \sum_{\substack{j=0\\2|j}}^{i} \cdots \sum_{\substack{m=0\\2|m}}^{s} (\frac{m}{2}+1) = \binom{\frac{\nu}{2}+r-2}{r-2},$$

here $k = \nu$ is even. Thus, we have proved the following

Theorem 2. *The maximal number of linearly independent theta-series with the spherical polynomial P of order ν for the diagonal quadratic form Q of r variables is $\binom{\frac{\nu}{2}+r-2}{r-2}$ and the basis of the space $T(\nu, Q)$ is among the theta-series $\vartheta(\tau, P, Q)$ with the spherical polynomial $P = P_{kij\ldots l}$ with even indices $k, i, j, \ldots l$.*

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Received 30.05.2016; revised 27.11.2016; accepted 20.11.2016.

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