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THE CONVERGENCE OF AN ITERATION METHOD FOR THE PLATE UNDER THE ACTION OF A SYMMETRIC LOAD

Jemal Peradze

Abstract. We consider a boundary value problem for a Timoshenko system of nonlinear ordinary differential equations describing the plate static behavior. The unknown functions u, w and ψ are the longitudinal and the transverse displacement and the angle of rotation of the normal of the plate. The functions u and ψ are expressed explicitly through the function w for which a nonlinear integro-differential equation with a boundary condition is written. To approximate the problem solution for w, the Galerkin method is used. It leads to a nonlinear system of algebraic equations that is solved by the Jacobi iteration method. The convergence of the iteration method is established and the error estimate is obtained.

Keywords and phrases: Nonlinear Timoshenko system, iteration method, error estimate.

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1 Introduction. If from the system of Timoshenko equations for a shell given in [2, p. 42], we discard the variables t and y and assume $k_x = k_y = 0$, we obtain a one-dimensional system of equations which characterizes the static state of the plate under the action of axially symmetric load. It has the form

$$\begin{aligned}
&u'' + \frac{1}{2}\left(w'^2\right)' + p(x) = 0,\\
&k_0^2\,\frac{Eh}{2(1+\nu)}\,(w'' + \psi') + \frac{Eh}{1-\nu^2}\left[\left(u' + \frac{1}{2}\,w'^2\right) w'\right]' + q(x) = 0,\\
&\frac{h^2}{6(1-\nu)}\,\psi'' - k_0^2\,(w' + \psi) = 0, \quad 0 < x < 1.
\end{aligned} \tag{1}$$

Suppose the following boundary conditions are fulfilled

$$u(0) = u(1) = 0, \quad w(0) = w(1) = 0, \quad \psi'(0) = \psi'(1) = 0. \tag{2}$$

Here the displacements $u = u(x)$, $w = w(x)$ of the plate midplane and the angle of rotation $\psi = \psi(x)$ of the normal to the midplane are the unknown functions to be determined, whereas the forces $p(x)$ and $q(x)$ are the given ones. E is Young's modulus, h is the plate thickness, k_0 is the lateral shear coefficient and ν is the Poisson ratio, $0 < \nu < 0.5$. Note that these equations can also be obtained from Timoshenko equations for a plate in [1, p. 24]. Using the first and the third equation from (1) and taking into account the respective boundary conditions from (2), the functions $u(x)$ and $\psi(x)$ can be expressed through the function $w(x)$ as follows

$$\begin{aligned} u(x) &= \int_0^1 G_u(x,\xi) w'^2(\xi)\, d\xi + \int_0^1 G_p(x,\xi) p(\xi)\, d\xi, \\ \psi(x) &= \int_0^1 G_\psi(x,\xi) w'(\xi)\, d\xi, \end{aligned} \tag{3}$$

where

$$G_u(x,\xi) = \begin{cases} \frac{1}{2}(x-1), & x > \xi, \\ \frac{1}{2}x, & x < \xi, \end{cases} \qquad G_p(x,\xi) = \begin{cases} \xi(1-x), & x > \xi, \\ x(1-\xi), & x < \xi, \end{cases}$$

$$G_\psi(x,\xi) = \begin{cases} -\dfrac{\sigma}{\sinh\sigma} \cosh\sigma(x-1)\cosh\sigma\xi, & x > \xi, \\ -\dfrac{\sigma}{\sinh\sigma} \cosh\sigma x \cosh\sigma(\xi-1), & x < \xi, \end{cases}$$

and $\sigma = \frac{k_0}{h}\sqrt{6(1-\nu)}$. Applying (3) in the second equation of system (1), we come to the equation for the function $w(x)$

$$\begin{aligned} &\frac{Eh}{1-\nu^2}\left[\left(\frac{1-\nu}{2}k_0^2 + \frac{1}{2}\int_0^1 w'^2 dx + \int_0^1 (1-x)p(x)\,dx - \int_0^x p(\xi)\,d\xi\right) w'' - p(x)w'\right] \\ &\quad - \frac{3Ek_0^4}{h\sinh\sigma}\,\frac{1-\nu}{1+\nu}\left(\sinh\sigma(x-1)\int_0^x \cosh\sigma\xi\, w'(\xi)\,d\xi\right. \\ &\quad \left. + \sinh\sigma x \int_x^1 \cosh\sigma(\xi-1) w'(\xi)\,d\xi\right) + q(x) = 0, \end{aligned} \tag{4}$$

to which we add the boundary condition from (2)

$$w(0) = w(1) = 0. \tag{5}$$

Thus problem (1), (2) reduces to problem (4), (5) for the function $w(x)$. After solving the latter problem, we construct the functions $u(x)$ and $\psi(x)$ by explicit formulas of form (3).

Now let us consider the question of approximate solution of problem (4), (5).

The approximation of $w(x)$ is written as the finite sum

$$w_n(x) = \sum_{i=1}^n \frac{1}{i\pi} w_{ni} \sin i\pi x, \tag{6}$$

where, in case we use the Galerkin method, the coefficients w_{ni} satisfy the nonlinear system of equations

$$\left(p_{1i} + p_2 + \sum_{j=1}^n w_{nj}^2\right) w_{ni} + \sum_{j=1}^n p_{3ij} w_{nj} + \frac{1}{i} q_i = 0, \quad i = 1, 2, \ldots, n. \tag{7}$$

Here the following notation is used

$$p_{1i}=\frac{1}{\frac{1}{2k_0^2(1-\nu)}+\frac{3}{(h\pi i)^2}},\quad p_2=4\int_0^1(1-x)p(x)\,dx,$$

$$p_{3ij}=-8\int_0^1\left(\int_0^x p(\xi)d\xi\right)\cos i\pi x\cos j\pi x\,dx,\quad q_i=-\frac{8(1-\nu^2)}{Eh\pi}\int_0^1 q(x)\sin i\pi x\,dx. \tag{8}$$

We will solve system (7) by using the Jacobi iteration method

$$\left(p_{1i}+p_2+w_{ni,k+1}^2+\sum_{\substack{j=1\\ j\neq i}}^n w_{nj,k}^2\right)w_{ni,k+1}+p_{3ii}w_{ni,k+1}+\sum_{\substack{j=1\\ j\neq i}}^n p_{3ij}w_{nj,k}+\frac{1}{i}q_i=0, \tag{9}$$

$$k=0,1,\dots\,,\quad i=1,2,\dots,n,$$

where $w_{ni,k+l}$ is the $(k+l)$-th approximation of w_{ni}, $l=0,1$. To realize iteration (9), we have to solve a cubic equation with respect to $w_{ni,k+1}$. Therefore, using the Cardano formula $w_{ni,k+1}$ can be written in the explicit form

$$w_{ni,k+1}=\sigma_{i,1}-\sigma_{i,2},\quad k=0,1,\dots,\quad i=1,2,\dots,n, \tag{10}$$

where

$$\sigma_{i,l}=\left[(-1)^l\frac{s_i}{2}+\left(\frac{s_i^2}{4}+\frac{r_i^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},\quad l=1,2,$$

$$r_i=p_{1i}+p_2+p_{3ii}+\sum_{\substack{j=1\\ j\neq i}}^n w_{nj,k}^2,\quad s_i=\frac{1}{i}\,q_i+\sum_{\substack{j=1\\ j\neq i}}^n p_{3ij}w_{nj,k}\,.$$

Let us estimate the error of the iteration process of the function $w(x)$. By this error we mean the difference between the function (7) and the function $w_{n,k}(x)=\sum_{i=1}^n\frac{1}{i\pi}\,w_{ni,k}\sin i\pi x$, obtained during the realization of the iteration process (10), i.e. the function $w_n(x)-w_{n,k}(x)$.

Theorem. *Suppose that the functions $p(x)$, $q(x)$ and constants ν. E, h, k_0 are such that*

$$\left(\frac{1}{2k_0^2(1-\nu)}+\frac{3}{(hi\pi)^2}\right)^{-1}>\left|4\int_0^1(1-x)p(x)\,dx\right.$$
$$\left.-8\int_0^1\left(\int_0^x p(\xi)\,d\xi\right)\cos^2 i\pi x\,dx\right|,\ i=1,2,\dots,n\,, \tag{11}$$

and

$$\frac{1}{2}\max_{1\le j\le n}\sum_{\substack{i=1\\ i\neq j}}^n\frac{1}{c_i}\,|p_{3ij}|+\frac{1}{2}\sum_{i=1}^n\frac{1}{c_i}\left(2\,\frac{|q_i|}{i\sqrt{c_i}}+\left(\sum_{\substack{j=1\\ j\neq i}}^n p_{3ij}^2\right)^{\frac{1}{2}}\right)<\Delta<1. \tag{12}$$

Then there exists a unique solution w_{ni}, $i = 1, 2, \dots, n$, *of system* (7) *to which the sequence of approximations* $w_{ni,k}$ *of the iteration method converges as* $k \to \infty$. *For the error of the iteration method we have*

$$\left\| \frac{d^l}{dx^l} \, (w_n(x) - w_{n,k}(x)) \right\|_{L_2(0,1)} \leq \frac{\Delta^k}{\sqrt{2}\,\pi^{1-l}(1-\Delta)} \sum_{i=1}^{n} |w_{ni,1} - w_{ni,0}|, \ l=0,1, \ k=0,1,\dots.$$

Note that instead of requirements (11) and (12) we can use more rigid but easily verifiable conditions

$$c = \Big(\frac{1}{k_0^2(1-\nu)} + \frac{6}{(h\pi)^2} \Big)^{-1} - \Big(\frac{5}{\sqrt{3}} + \frac{\sqrt{2}}{\pi} \Big) p_0 > 0 \tag{13}$$

and

$$\frac{1}{2c} \left(\Big(\frac{1}{\pi} + \left(\frac{1}{\sqrt{2}} + \frac{4}{\pi} \right) n + \sqrt{2}(n-1)n \Big) p_0 + 8 \sqrt{\frac{2}{c} \frac{1-\nu^2}{Eh\pi}}\, q_0 \right) \sum_{i=1}^{n} \frac{1}{i} < \Delta < 1, \tag{14}$$

where $p_0 = \left(\int_0^1 p^2(x)\, dx \right)^{\frac{1}{2}}$, $q_0 = \left(\int_0^1 q^2(x)\, dx \right)^{\frac{1}{2}}$. The conditions (11) and (12) as well as (13) and (14) are fulfilled for sufficiently small functions $p(x)$ and $q(x)$.

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Author(s) address(es):

Jemal Peradze
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: j_peradze@yahoo.com

Georgian Technical University
Kostava str. 77, 0174 Tbilisi, Georgia
E-mail: j_peradze@yahoo.com