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ON THE APPROXIMATE SOLUTION OF THE J. BALL NONLINEAR DYNAMIC BEAM EQUATION ⋆

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Abstract. An initial boundary value problem for a J. Ball nonlinear dynamic beam equation is considered. For approximate solution of the problem projection method, symmetrical difference scheme and iteration process have been used. The accuracy of the algorithm is studied.

Keywords and phrases: Nonlinear beam equation, Galerkin method, method error.

AMS subject classification (2010): 65M60, 65N15.

1 Statement of the problem. Let us consider the initial boundary value problem

$$\begin{gathered} u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t) \\ -\left(\beta+\rho\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t) \\ -\sigma\left(\int_0^L u_x(x,t)u_{xt}(x,t)\,dx\right)u_{xx}(x,t)=0, \end{gathered} \tag{1}$$

$$\begin{gathered} 0<x<L, \quad 0<t\le T, \\ u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \\ u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0, \end{gathered} \tag{2}$$

where α, γ, ρ, σ, β and δ are the given constants among which the first four are positive numbers, while $u^0(x)$ and $u^1(x)$ are the given functions.

The equation (1) obtained by J. Ball [1] using the Timoshenko [5] theory describes the vibration of a beam. The problem of construction of an approximate solution for this equation is dealt with in [2] - [4].

2 Galerkin method and its error. We write an approximate solution of the problem (1), (2) in the form $u_n(x,t)=\sum\limits_{i=1}^{n}u_{ni}(t)\sin\dfrac{i\pi x}{L}$, where the coefficients $u_{ni}(t)$ will

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be found by the Galerkin method from the system of equations

$$\begin{gathered} u''_{ni}(t) + \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right) u'_{ni}(t) + \left[\alpha\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2\left(\beta + \rho\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2(t)\right.\right. \\ \left.\left. +\sigma\,\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}(t)u'_{nj}(t)\right)\right]u_{ni}(t) = 0, \\ i = 1, 2, \ldots, n, \quad 0 < t \le T, \end{gathered} \tag{3}$$

with the initial conditions

$$u_{ni}(0) = a_i^0, \quad u'_{ni}(0) = a_i^1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Here a_i^0 and a_i^1 are the coefficients from the representation of the functions $u^0(x)$ and $u^1(x)$ as $u^p(x) = \sum\limits_{i=1}^{\infty} a_i^p \sin\dfrac{i\pi x}{L}$, $p = 0, 1$.

If

$$u^p(x) \in L^2(0, L), \quad p = 0, 1, \tag{5}$$

then there exists a generalized solution of the problem (1), (2) that is a function $u(x,t)$ representable as a series $\sum\limits_{i=1}^{\infty} u_i(t) \sin\dfrac{i\pi x}{L}$, the coefficients of which satisfy the system of equations

$$\begin{gathered} u''_i(t) + \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right) u'_i(t) + \left[\alpha\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2\left(\beta + \rho\,\frac{L}{2}\sum_{j=1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j^2(t)\right.\right. \\ \left.\left. +\sigma\,\frac{L}{2}\sum_{j=1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j(t)u'_j(t)\right)\right]u_i(t) = 0, \\ i = 1, 2, \ldots, \quad 0 < t \le T, \end{gathered} \tag{6}$$

with the initial conditions

$$u_i(0) = a_i^0, \quad u'_i(0) = a_i^1, \quad i = 1, 2, \ldots . \tag{7}$$

Denote $\Delta u_{ni}(t) = u_{ni}(t) - u_i(t)$ and assume that under the method error we will understand the function $\Delta u_n(x,t) = \sum\limits_{i=1}^{n} \Delta u_{ni}(t) \sin\dfrac{i\pi x}{L}$ the $L^2(0,L)$-norm of which we want to estimate.

Subtract (6) from (3) and, having multiplied the resulting equality by $2\Delta u'_{ni}(t)$, sum it over $i=1,2,\ldots,n$. We obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{n}\Delta u_{ni}'^{2}(t)+2\sum_{i=1}^{n}\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^{4}\right)\Delta u_{ni}'^{2}(t)\\
&\quad+\frac{d}{dt}\sum_{i=1}^{n}\left(\alpha\left(\frac{i\pi}{L}\right)^{4}+\beta\left(\frac{i\pi}{L}\right)^{2}\right)\Delta u_{ni}^{2}(t)\\
&+\rho\,\frac{L}{2}\Bigg[\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^{2}(u_{nj}(t)+u_{j}(t))\Delta u_{nj}(t)\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^{2}(u_{ni}(t)+u_{i}(t))\Delta u_{ni}'(t)\\
&\quad+\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^{2}(u_{nj}^{2}(t)+u_{j}^{2}(t))\,\frac{d}{dt}\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^{2}\Delta u_{ni}^{2}(t)\Bigg]\\
&+\frac{\sigma L}{4}\Bigg\{\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^{2}\Big[(u_{nj}'(t)+u_{j}'(t))\Delta u_{nj}(t)+(u_{nj}(t)+u_{j}(t))\Delta u_{nj}'(t)\Big]\\
&\quad\times\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^{2}(u_{ni}(t)+u_{i}(t))\Delta u_{ni}'(t)\\
&+\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^{2}(u_{nj}(t)u_{nj}'(t)+u_{j}(t)u_{j}'(t))\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^{2}\frac{d}{dt}\,\Delta u_{ni}^{2}(t)\Bigg\}\\
&=L\left(\rho\sum_{j=n+1}^{\infty}\left(\frac{j\pi}{L}\right)^{2}u_{j}^{2}(t)+\sigma\sum_{j=n+1}^{\infty}\left(\frac{j\pi}{L}\right)^{2}u_{j}(t)u_{j}'(t)\right)\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^{2}\Delta u_{ni}'(t)u_{i}(t).
\end{aligned}$$

Subtracting (7) from (4) for $i=1,2,\ldots,n$, we have

$$\Delta u_{ni}(0)=0,\quad \Delta u_{ni}'(0)=0,\quad i=1,2,\ldots,n.$$

Our further consideration will be restricted to a more difficult case when β and δ are negative numbers. For three other combinations of these numbers we come to a result analogous to the one given at the end of this paper.

Multiply the equation (3) by $2u_{ni}'(t)$ and the equation (6) by $2u_i'(t)$. Then sum the obtained equalities over $i=1,2,\ldots,n$ in the first case, and over $i=1,2,\ldots$ in the second case. As a result, after some transformations, we make the following conclusion.

Theorem. *If the requirement* (5) *is fulfilled for the functions $u^p(x)$, $p=0,1$, and the above-mentioned conditions are fulfilled for the constants α, γ, ρ, σ, β and δ, then for the Galerkin method error the estimate*

$$\left\|\frac{\partial}{\partial t}\,\Delta u_n(x,t)\right\|_{L^2(0,L)}^2+\alpha\left\|\frac{\partial^2}{\partial x^2}\,\Delta u_n(x,t)\right\|_{L^2(0,L)}^2\le c\left(\sum_{i=n+1}^{\infty}a_i^{1\,2}+\sum_{i=n+1}^{\infty}\left(\frac{i\pi}{L}\right)^4 a_i^{0\,2}\right)^2$$

holds, where

$$c=e^{\frac{1}{2}T\left(\nu-2\delta+\left((\nu-2\delta)^2+\frac{\nabla^2}{\alpha}\right)^{\frac{1}{2}}\right)}.$$

Remark. The algorithm of nonlinear dynamic beam's approximate solution additionally consist of two steps - symmetrical difference scheme and iteration process. For approximate solution of (3)-(4) problems we are using difference scheme and Jacob's iteration method. Also for solution of the nonlinear equation we are using well-known Cardano's formula. We have estimated error of the iteration method by L^2 norm.

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