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THE NUMERICAL SOLUTION FOR A TWO-POINT BOUNDARY VALUE PROBLEM WITH A NON-CONSTANT COEFFICIENT BY MEANS OF OPERATOR INTERPOLATION METHOD

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Abstract. The new numerical algorithms for a two-point boundary value problem with a non-constant coefficient are proposed. The Green function of the given problem is represented as a nonlinear operator with respect to the coefficient. This operator is approximated by an operator interpolation polynomial of the Newton type. For the inverse operators approximate formulas of different types are derived. The numerical algorithms and results of calculation of test problems are given.

Keywords and phrases: Two-point boundary value problem, Green function, operator interpolation polynomial of the Newton type.

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1 Introduction. functional series and interpolation polynomials for solving identification problems are used in the theory of nonlinear systems. The interpolation formula of the Newton type is constructed and evaluation of residual term in V. Makarov's and V. Khlobistov's works for nonlinear operator's is obtained (see for example [1], [2]). This approach is based on "continual" knots from interpolation conditions in the definition of kernels of operator polynomials. These "continual" knots represent the linear combination of Heavyside functions. The above mentioned works have theoretical and practical importance in applied problems of the theory of operators' approximation. Issues of realization of interpolation approximations on the electronic computers haven't been discussed by the above mentioned authors. Calculating algorithms for approximate solution for boundary value problems of elliptic differential equations with non-constant coefficients are desscribed in the works [3], [4], results of calculations of test problems are given, convergence issues are studied in the numerical-experimental way.

Issues of approximate solutions for the two-point boundary value problem with non-constant coefficient by the use of operator interpolation polynomials of the Newton type are also discussed in the given work. Besides, the Green function of the differential equation(of the boundary value problem)as a non-linear operator with respect to the non-constant coefficient, is replaced by the known kernels of operator interpolation polynomial of the Newton type. Formulas of approximate solution of different types are constructed for finding the solution for two-point boundary value problem. Description of realization algorithm and the calculation results of test problems are given. The convergence with respect to m parameter from the series of numerical experiments is exposed (m - degree of the operator interpolation polynomial of the Newton type).

2 Statement of the problem. Let's consider be two-point boundary value problem for a differential equation

$$\begin{cases} u"(x) - q(x)u(x) = -f(x) \qquad x \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

$$q(x) \geq 0, \quad q(x), f(x) \in L_2[0,1].$$

problem (1) has the unique solution in $W^2_{2,0}(0,1)$ space, that can be represented by the use of Green's function in the following way:

$$u(x) = \int_0^1 G(x,\xi,q(\cdot))f(\xi)d\xi. \tag{2}$$

Notation $G(x,\xi,q(\cdot))$ shows that $G(x,\xi)$ Green's function of the given boundary value problem depends on $q(x)$ function (non-constant coefficient), so that Green's function can be considered as a nonlinear coefficient with respect to q

$$G : L_2^+[0,1] \to C([0,1]\times[0,1]),$$

$$L_2^+[0,1] = \Big\{ q(x) : q(x) \in L_2[0,1], \;\; q(x) \geq 0, \quad \forall x \in [0,1] \Big\}.$$

Let's consider the operator-interpolation approach of the solution of (1) boundary value problem. Replace $G(x,\xi,q(\cdot))$ Green's function of the boundary value problem by m order operator interpolation polynomial of the Newton type in the following way:

$$G_m(x,\xi,q(\cdot)) = G(x,\xi,0) +$$

$$+\sum_{i=1}^{m} \int_0^1 \cdots \int_0^1 \Big\{ K_i(x,\xi,z_1,\cdots,z_i) \prod_{j=1}^{i} H(z_j - z_{j-1}) \cdot \Big[q(z_j) - h(j-1) \Big] \Big\} dz_1 \cdots dz_i, \tag{3}$$

$$G(x,\xi,0) = \begin{cases} x(1-\xi), & 0 \leq x \leq \xi, \\ \xi(1-x), & \xi \leq x \leq 1, \end{cases} \qquad z_0 = 0,$$

where the operator kernels $K_i(x,\xi,z_1,\cdots,z_i) = \dfrac{(-1)^1}{h^1} \dfrac{\partial^i}{dz_1 \cdots dz_i} G(x,\xi,\zeta_i)$,

$$\zeta_i = h \sum_{j=1}^{i} H(\cdot - z_j), \quad i = 1,2,\cdots,m.$$

Heavyside's function

$$H(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0, \end{cases}$$

step of the interpolation net

$$h = \frac{(c_m - c_0)}{(m+1)}, \quad 0 \leq q(x) \leq c, \quad c_0 = \min_{x\in[0,1]} q(x),$$

$$c_m = \max_{x\in[0,1]} q(x), \quad q(x) \in C_{[0,1]}.$$

3 The algorithm. The algorithm consists of two main parts: 1. construction of the operator kernels, 2. construction of the approximate solution for the boundary value problem.

3.1 First part. Construction of the operator kernels.

$K_1, K_2, \cdots$ Operator kernels, defined by Green's function and constructed by the use of main properties (see for example [5]), have the following image:

$$K_1(x,\xi,z_1) = -G(x,z_1,\zeta_i) * G(z_1,\xi,\zeta_i),$$

$$K_2(x,\xi,z_1,z_2) = G(x,z_1,\zeta_2) * G(z_1,z_2,\zeta_2) * G(z_2,\xi,\zeta_2)$$

$$+G(x,z_2,\zeta_2) * G(z_2,z_1,\zeta_2) * G(z_1,\xi,\zeta_2),$$

and etc. Construction of high order operator kernels is labour-consuming, on the other hand construction of the operator kernels for the given boundary value problem is carried out once and for all. Operator kernel, depended on $\xi, z_1, \cdots, z_i$ parameters, represents the sum of $i!$ number of summands of the product of $i+1$ Green's function. Note that, when using this method, by the increase of approximating polynomial's order, all the kernels calculated on the previous step, remain unchanged. The increase of approximation order, besides the addition of new terms, causes decrease of the step of only the image's net.

3.2 Second part. Construction of the approximate solution for the boundary value problem. The m-ary approximate solution $u_m(x)$ of (1) boundary value problem has the following form:

$$u_m(x) = \int_0^1 G_m(x,\xi,q(\cdot))f(\xi)d\xi, \quad m = 0,1,2,\cdots,$$

$$G_0(x,\xi,q(\cdot)) = G(x,\xi,0),$$

where $G_m(x,\xi,q(\cdot))$ is found by formula (3). To find $u_0(x)$ zero approximation, we need to calculate two unitary integrals, for $u_1(x)$ - 6 double integrals, for $u_2(x)$ - 12 triple integrals, etc. In the general case, in order to calculate $u_i(x)$ we need to find $(i+1)(i+2)$ number of the sum of $(i+1)$ - multiples integrals. As it was mentioned above, by increasing m, significantly increases capacity of work. In the constructed formulas, $f(x)$ and $q(x)$ are parameters.

4 Numerical experiments and results of calculation. For the approximate solution of the given boundary value problem, the software product is compiled on the algorithmic language MATLAB and is realized on a computer. For the various test problems, numeral results are obtained for the zero, the first and the second approximation of the solution.

Let's consider the following test problem with the non-constant coefficient $q(x) = 1 + x^2$, the right side $f(x) = (1 + \pi^2 + x^2)\sin \pi x$, exact solution $u(x) = \sin \pi x$.

x/u	$u_{exact}(x)$	$u_0(x)$	error	$u_1 x)$	error	$u_2(x)$	error
0.25	0.70711	0.79473	0.08762	0.69647	0.01064	0.70456	0.00254
0.50	1.00000	1.12956	0.12956	0.95838	0.04162	0.97554	0.02446
0.75	0.70711	0.80384	0.09673	0.67535	0.03176	0.68856	0.01854

Remark. When the solution of the boundary value problem is a fast oscillated function, then operating interpolation method gives the better results than finite difference-method.

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