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THE PROBLEM OF EXISTENCE THE NEUTRAL SURFACE FOR THE ELASTIC SHELL *

Tengiz Meunargia

Abstract. I. Vekua obtained the conditions for the existence of the neutral surface of a shell, when the neutral surface is the middle surface. In this paper the neutral surface is considered as any equidistant surfaces of the shell.

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The stress-strain relations are given in the form [1, 2]

$$\sigma_j^i = \lambda\theta g_j^i + 2\mu e_j^i, \quad (i, j = 1, 2, 3), \tag{1}$$

where σ_j^i and e_j^i are the mixed components, respectively of stress and strain tensors, θ is the cubical dilatation which will be written as

$$\theta = \theta' + e_3^3, \quad \theta' = e_\alpha^\alpha, \quad (\alpha = 1, 2). \tag{2}$$

when $j = 3$ from (1) we have

$$\sigma_3^\alpha = 2\mu e_3^\alpha, \quad \sigma_3^3 = \lambda\theta + 2\mu e_3^3 = \lambda\theta' + (\lambda + 2\mu) e_3^3. \tag{3}$$

From (3)

$$e_3^\alpha = \frac{1}{2\mu}\sigma_3^\alpha, \quad e_3^3 = -\frac{\lambda}{\lambda + 2\mu}\theta' + \frac{1}{\lambda + 2\mu}\sigma_3^3. \tag{4}$$

By inserting (4) into (2) we obtain

$$\theta = \frac{\lambda'}{\lambda}\theta' + \frac{1}{\lambda + 2\mu}\sigma_3^3, \quad \lambda' = \frac{2\lambda\mu}{\lambda + 2\mu}. \tag{5}$$

Substituting expression (5) into (1) we get

$$\sigma_j^i = T_j^i + Q_j^i = \left(\lambda'\theta' + \frac{\lambda}{\lambda + 2\mu}\sigma_3^3\right) g_j^i + 2\mu e_j^i,$$

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where

$$T^{\alpha}_{\beta}=\lambda'\theta g^{\alpha}_{\beta}+2\mu e^{\alpha}_{\beta},\ Q^{\alpha}_{\beta}=\sigma'\sigma^3_3 g^{\alpha}_{\beta},\ T^i_3=0,\ Q^i_3=\sigma^i_3,\ \Big(\sigma'=\frac{\lambda}{\lambda+2\mu}\Big). \tag{6}$$

The vector $\boldsymbol{T}^{\alpha}$ satisfies the condition $\boldsymbol{n}\boldsymbol{T}^{\alpha}=0$ and is therefore called the tangential stress field and the vector $\boldsymbol{Q}^i$ will be called the transverse field.

The vectorial equation of equilibrium

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{\sigma}^i)+\boldsymbol{\Phi}=0,\quad (\sqrt{g}=\sqrt{a}\vartheta,\ \ \vartheta=1-2Hx_3+Kx_3^2) \tag{7}$$

may be written as

$$\frac{1}{\sqrt{g}}[\partial_{\alpha}(\sqrt{g}\boldsymbol{T}^{\alpha})+\partial_i(\sqrt{g}\boldsymbol{Q}^i)]+\boldsymbol{\Phi}=0. \tag{8}$$

Let the surface $\hat{S}:x^3=\text{const}$ be the neutral surface of a non-shallow shell. Then $\boldsymbol{T}^{\alpha}=0$, i.e. $T^{\alpha\beta}=0$ (on $\hat{S}$), and equation (8)

$$\frac{1}{\sqrt{a}}\partial_{\alpha}(\sqrt{a}\vartheta\boldsymbol{Q}^{\alpha})+\partial_3(\vartheta\boldsymbol{\sigma}^3)+\vartheta\boldsymbol{\Phi}=0,$$

or

$$\big[\nabla_{\alpha}(\vartheta\boldsymbol{Q}^{\alpha})+\partial_3(\vartheta\boldsymbol{\sigma}^3)+\vartheta\boldsymbol{\Phi}\big]_{x^3=\text{const}}=0,\quad (-h\le x^3=x_3\le h) \tag{9}$$

where $2h$ is the thickness of the shell and

$$\boldsymbol{Q}^{\alpha}=\sigma'\sigma^3_3\boldsymbol{R}^{\alpha}+\sigma^{\alpha}_3\boldsymbol{n},\ \ \nabla_{\alpha}(\cdot)=\frac{1}{\sqrt{a}}\partial_{\alpha}(\sqrt{a}\ \cdot). \tag{10}$$

Denote the stress forces acting on the face surfaces S^+ and S^- by $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$. We have

$$\overset{(+)}{\boldsymbol{P}}=-(\boldsymbol{\sigma}^3)_{x^3=h},\ \ \overset{(-)}{\boldsymbol{P}}=(\boldsymbol{\sigma}^3)_{x^3=-h}. \tag{11}$$

If we approximately represent $\boldsymbol{\sigma}^3$ by the formula

$$\boldsymbol{\sigma}^3(x^1,x^2,x^3)\cong\overset{0}{\boldsymbol{\sigma}}(x^1,x^2)+x^3\,\overset{1}{\boldsymbol{\sigma}}(x^1,x^2). \tag{12}$$

from (11) we get

$$\begin{aligned}\boldsymbol{\sigma}^3(x^1,x^2,x^3)&\cong-\frac{1}{2}\Big[\overset{(+)}{\boldsymbol{P}}-\overset{(-)}{\boldsymbol{P}}+\frac{x^3}{h}\Big(\overset{(+)}{\boldsymbol{P}}+\overset{(-)}{\boldsymbol{P}}\Big)\Big]\\&=-\frac{1}{2}\Big[\frac{h+x_3}{h}\Big(\overset{(+)}{\boldsymbol{P}}-\overset{(-)}{\boldsymbol{P}}\Big)+\frac{2x^3}{h}\overset{(-)}{\boldsymbol{P}}\Big],\end{aligned} \tag{13}$$

or

$$\boldsymbol{\sigma}^3(x^1,x^2,x^3)=-\frac{1}{2}\Big[\frac{h+x_3}{h}\Big(P^{\alpha}\mathbf{r}_{\alpha}+P^3\mathbf{n}\Big)+\frac{2x^3}{h}\overset{(-)}{\boldsymbol{P}}\Big], \tag{14}$$

$$P^\alpha = \overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha}, \quad P = \overset{(+)}{P^3} - \overset{(-)}{P^3}.$$

Then to define the vector field $\overset{(+)}{\boldsymbol{P}}$ we have the equation

$$\left\{\nabla_\alpha(\sigma' A^\alpha_\beta P \boldsymbol{r}^\beta + A P^\alpha \boldsymbol{n}) + B(P\boldsymbol{n} + P^\alpha \boldsymbol{r}_\alpha) + \tilde{\boldsymbol{\Phi}}\right\}_{x^3=c} = 0 \tag{15}$$

where

$$\begin{aligned}
&A^\alpha_\beta = \frac{h+c}{h}[a^\alpha_\beta + c(b^\alpha_\beta - 2Ha^\alpha_\beta)], \quad A = \frac{h+c}{h}\vartheta(c),\\
&B = \frac{1}{h}[1 - 2Hh + 2(Kh - 2H)c + 3Kc^2],\\
&\tilde{\boldsymbol{\Phi}} = -2\vartheta(c)\boldsymbol{\Phi}(c) + \nabla_\alpha\left\{\sigma'\frac{2c}{h}\left[a^\alpha_\beta + c(b^\alpha_\beta - 2Ha^\alpha_\beta)\right]\overset{(-)}{P^3}\boldsymbol{r}^\beta + \frac{2c}{h}\vartheta(c)\overset{(-)}{P^\alpha}\boldsymbol{n}\right\}\\
&+\frac{2}{h}\left[\vartheta(c) + 2(Kc - H)\right]\overset{(-)}{\boldsymbol{P}}.
\end{aligned} \tag{16}$$

Since

$$\nabla_\alpha \boldsymbol{r}^\beta = b^\beta_\alpha \boldsymbol{n}, \text{ and } \nabla_\alpha \boldsymbol{n} = -b_{\alpha\beta}\boldsymbol{r}^\beta$$

from (15) we have

$$\sigma'\nabla_\alpha(A^\alpha_\beta P) + (Ba_{\alpha\beta} - Ab_{\alpha\beta})P^\alpha + \tilde{\Phi}_\beta = 0, \ \ (\tilde{\Phi}_\beta = \tilde{\boldsymbol{\Phi}}\boldsymbol{r}_\beta), \tag{17}$$

$$\nabla_\alpha(AP^\alpha) + (\sigma' A^\alpha_\beta b^\beta_\alpha + B)P + \tilde{\Phi}_3 = 0, \ \ (\tilde{\Phi}_3 = \tilde{\boldsymbol{\Phi}}\boldsymbol{n}). \tag{18}$$

From the system of equations (17) we have

$$P^\alpha = \overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha} = -\hat{d}^{\alpha\beta}\left[\nabla_\alpha(A^\alpha_\beta P) + \tilde{\Phi}_\beta\right], \tag{19}$$

where

$$\begin{aligned}
&\hat{d}^{\alpha\beta} = \tfrac{1}{\Delta}\left[(B - 2AH)a^{\alpha\beta} + Ab^{\alpha\beta}\right], \ \ \hat{F}_\beta = -\left[\tilde{\Phi}_\beta + \nabla_\alpha(A^\alpha_\beta P)\right],\\
&\Delta = B^2 - 2ABH + A^2K.
\end{aligned} \tag{20}$$

Inserting expressions (19) into (18) we obtain the equation

$$\sigma'\nabla_\alpha\left[A\hat{d}^{\alpha\beta}\nabla_\gamma(A^\alpha_\beta P)\right] - (B + \sigma' A^\alpha_\beta b^\beta_\alpha)P + \Phi = 0. \tag{21}$$

It is easily seen that equation (21) is of the elliptic type.

Thus, if the surface $x^3 = c$ is neutral, the stress $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(+)}{\boldsymbol{P}}$, applied to the face surfaces, must satisfy the vector equation (17) and (18). This means that the stresses $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$ cannot be prescribed arbitrarily both at the same time. However there are problems when this does not occur. For example, in aircraft or submarine apparatus the force $\overset{(-)}{\boldsymbol{P}}$

acting on the inner surface S^- may be assumed to be prescribed, but the $\overset{(+)}{\boldsymbol{P}}$ acting on the external face surface S^+ is not, in general, assigned beforehand. The same situation occurs on dams. One face surface of the dam is free from stresses and the other is under the hydrodynamic load, a variable which is generally difficult to define exactly at any moment in time.

R E F E R E N C E S

1. Vekua, I. Shell Theory: General Methods of Construction, Pitman Advanced Publishing Program. *Boston-London-Melburne*, 1985.

2. Meunargia, T. Neutral surfaces of a non-shallow shells. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.*, **29** (2015), 88-91.

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Author(s) address(es):

Tengiz Meunargia
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tengiz.meunargia@viam.sci.tsu.ge